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SECOND COURSE IN ALGEBRA

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PREFACE

In the present volume, which is intended as a sequel to the author's *First Course in Algebra*, the following features may be noted:

(*a*) The first seven chapters furnish a systematic review of the ordinary first course up to and including the subject of simultaneous linear equations. In these opening chapters the aim is to state briefly and concisely those fundamental principles which are basal in all algebraic work and with which the pupil already has some acquaintance. The method of treatment is largely inductive, being based upon solved problems and other illustrative material rather than upon any attempt at proofs or formal demonstrations. The various principles are explicitly stated, however, at all points in the form of rules, which are clearly set off in italics. Upon this plan the pupil is rapidly and effectively prepared for undertaking the newer and more advanced topics which follow.

(*b*) Chapters VIII–IX (Square Root and Radicals) are essentially a reproduction of the corresponding chapters in the *First Course*, but all of the problems are new. This is true also of the early parts of Chapter X (Quadratic Equations). These are topics which usually present more than average difficulty; hence, even for pupils who have studied them carefully before, a complete treatment of them is desirable in the second course.

The introduction and use of tables of square and cube roots at this point (§ 43) is to be especially noted. It seems

clear that pupils should be made familiar with such tables at an earlier date than formerly. One strong reason for this is that a constantly increasing number of students pass directly from the high schools into technical pursuits where facility in the manipulation of tables of all kinds is especially desirable.

(*c*) Part II, comprising Chapters XI–XX, presents the usual topics of the advanced course. The order of arrangement follows, so far as possible, that of the difficulty of the various subjects, and the whole has been prepared with a view of introducing a relatively large number of simple illustrative examples drawn from nature and the arts. Throughout the development, however, due emphasis has been given to those fundamental disciplinary values which should be preserved in any course in mathematics.

Among the unusual features, it may be observed that the detailed consideration of exponents and radicals has been delayed until logarithms are about to be taken up. These topics in their extended sense have, in fact, but little to do with algebra until that time.

Again, the chapter on logarithms is unusually full and complete. All the essential features of this relatively difficult but increasingly important subject are presented in detail. In the past, much has ordinarily been left for the teacher to explain.

(*d*) Functions, Mathematical Induction (including the proof of the Binomial Theorem), and Determinants have been grouped together under the title Supplementary Topics. In fact, these subjects lie on the border line between the second course and the college course. Only the elements of each are taken up, but there is enough to show its important bearing in algebra and to pave the way for its further development in the college course. For example,

the study of functions is so presented that it at once amplifies material to be found in the earlier chapters of the book and brings in new material which is connected with the graphical study of the theory of equations. Thus it serves as an introduction to the latter subject as presented in the usual college texts.

As in the authors' other texts, a star (*) has been placed against certain sections that may be omitted if desired without destroying the continuity of the whole.

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LAPLACE

(*Pierre Simon Laplace,* 1749–1827)

Famous in mathematics for his researches, which were of a most advanced kind, and especially famous in astronomy for his enunciation of the Nebular Hypothesis. Interested also in physics and at various times held high political offices under Napoleon.

SECOND COURSE IN ALGEBRA

PART I. REVIEW TOPICS †

CHAPTER I

FUNDAMENTAL NOTIONS

1. Negative Numbers. In the *First Course in Algebra* it was shown how negative as well as positive numbers may be used, the one being quite as common as the other in everyday life.

Thus $+15°$ (or simply $15°$) means $15°$ *above* $0°$, while $-15°$ means $15°$ *below* 0. Similarly, $+\$25$ (or simply \$25) means a gain or asset of \$25, while $-\$25$ means a loss or debt of the same amount.

Negative numbers are compared with each other in much the same way as positive numbers. Thus, just as in arithmetic 4 is less than 5, 3 is less than 4, etc., until finally we say that 0 is less than 1, so we continue this idea in algebra by saying that -1 is less than 0, -2 is less than -1, etc.

The whole situation regarding the size of numbers is vividly brought out to the eye in the figure below:

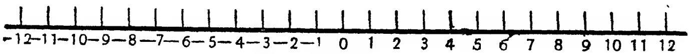


FIG. 1.

Here the positive (+) numbers are placed in their order to the *right* of the point marked 0, while the negative (−) numbers are placed in their order to the *left* of that same point. This figure shows *all* numbers (positive and negative) arranged in their increasing order from left to right.

† Chapters I–X (pp. 1–94) furnish a review of the *First Course*. The remaining chapters deal with more advanced topics.

In Fig. 1, only the positive and negative *integers* and zero are actually marked. A complete figure would show also the positions of the fractions. Thus $\frac{1}{2}$ is located at the point halfway between the 0 and the 1; $2\frac{1}{3}$ is located at the point one third the way from 2 to 3; $-\frac{3}{4}$ is at the point $\frac{3}{4}$ the distance from 0 to -1; $-5\frac{7}{8}$ is at the point $\frac{7}{8}$ the distance from -5 to -6; and so on for all fractions.

By the ***numerical value*** (or ***absolute value***) of a negative number is meant its corresponding positive value.

Thus the numerical, or absolute, value of -3 is $+3$, or simply 3.

NOTE. The numerical value of any positive number is the number itself.

EXERCISES

In each of the following exercises, state which of the two numbers is the larger. First locate the number at its proper place on the line shown in § 1.

1. 7, 10.	**5.** $\frac{2}{3}$, $\frac{3}{4}$.	**9.** $3\frac{1}{2}$, $1\frac{8}{9}$.
2. 7, -10.	**6.** $\frac{2}{3}$, $-\frac{3}{4}$.	**10.** $-3\frac{1}{2}$, $-1\frac{8}{9}$
3. -7, 10.	**7.** $-\frac{2}{3}$, $\frac{3}{4}$.	**11.** $-\frac{5}{6}$, 0.
4. -7, -10.	**8.** $-\frac{2}{3}$, $-\frac{3}{4}$.	**12.** $-.3$, $-.05$

2. Operations with Numbers. The following facts will be recalled from the *First Course in Algebra*.†

(*a*) *To add two numbers having* **like** *signs, add their absolute values* (§ 1) *and prefix the common sign.*

Thus $(-5)+(-6)=-11$.

(*b*) *To add two numbers having* **unlike** *signs, find the difference between their absolute values and prefix the sign of the one whose absolute value is the greater*

Thus $(+3)+(-5)=-2$

† References to the authors' *First Course in Algebra* are given in this book by page number.

NOTE. If we have more than two numbers to add together, as for example $(+4)+(-7)+(+5)+(-6)+(-1)$, the customary way is to add all the positive parts together, then all the negative parts, and finally to add the two results thus obtained. Thus, in the example just mentioned, the sum of the positive parts is $4+5=9$, while the sum of the negative parts is $(-7)+(-6)+(-1)=-14$. The final result sought is, therefore, $(+9)+(-14)=-5$.

(c) *To subtract one number from another, change the sign of the subtrahend and add the result to the minuend.*

Thus $(-7)-(-5)=(-7)+(+5)=-2$.

(d) *To multiply one number by another, find the product of their absolute values, and take it* ***positive*** *if the two numbers have the* ***same*** *sign, but* ***negative*** *if they have* ***unlike*** *signs.*

Thus $(+3)\cdot(+2)=+6$, and $(-3)\cdot(-2)=+6$; but $(-3)\cdot(+2)=-6$ and $(+3)\cdot(-2)=-6$.

(e) *To divide one number by another, find the quotient of their absolute values, and take it* ***positive*** *if the numbers have the* ***same*** *sign, but* ***negative*** *if they have* ***unlike*** *signs.*

Thus $(+8)\div(+2)=+4$, and $(-8)\div(-2)=+4$; but $(-8)\div(+2)=-4$, and $(+8)\div(-2)=-4$.

EXERCISES

Determine the value of each of the following indicated expressions.

1. $(+4)+(+7)$.
2. $(-4)+(+7)$.
3. $(+4)+(-7)$.
4. $(-4)+(-7)$.
5. $(+9)+(-27)$.
6. $(-32)+(+16)$.
7. $(-3)+(+2)+(-1)$.
8. $(+25)+(-32)+(-5)+(+12)$.
9. $(+7)-(+4)$.
10. $(+7)-(-4)$.
11. $(-7)-(+4)$.
12. $(-7)-(-4)$.
13. $(+34)-(-63)$.
14. $(-54)-(-32)$.

15. $(+6)-(-4)+(-2)-(-5)$.

[HINT. By (c) of § 2, this may first be changed into the form $(+6)+(+4)+(-2)+(+5)$.]

16. $(-3)-(+2)-(-3)+(-6)$.

17. $(-23)+(+32)-(-27)+(+4)-(-14)$.

18. $(+3)\cdot(-4)$.

19. $(-4)\cdot(+3)$.

20. $(+5)\cdot(+4)$.

21. $(-5)\cdot(-4)$.

22. $(+24)\div(-6)$.

23. $(-36)\div(+6)$.

24. $(-55)\div(-11)$.

25. $(\frac{2}{3})\div(-\frac{1}{2})$.

3. Use of Letters in Algebra. Algebra is distinguished from arithmetic not only because of its use of negative numbers, but also because of its general use of letters to represent numbers. This is useful in many ways. In particular, it enables us to solve problems in arithmetic which would otherwise be very difficult. The following facts and definitions will be recalled from the *First Course* in this connection.

The ***sum*** of any two numbers, as x and y, is represented by $x+y$.

The ***difference*** between any two numbers, as x and y (meaning the number which added to y gives x), is represented by $x-y$.

The ***product*** of any two numbers, such as x and y, is written in the form xy. It has the same meaning as $x\times y$, or $x\cdot y$. Either of the numbers thus multiplied together is called a ***factor*** of the product.

The ***quotient*** of x divided by y is expressed either by $x\div y$, or by $\frac{x}{y}$, or by x/y.

The product $x\cdot x$ is represented by x^2 and is read ***x square;*** similarly, $x\cdot x\cdot x$ is represented by x^3 and is read ***x cube***. More generally, $x\cdot x\cdot x\cdots$ to n factors is represented by x^n and is read ***x to the nth power***. The letter n as thus used in x^n is called the ***exponent*** of x.

The symbol $\sqrt{x}$ denotes that number which when squared gives x. It is called ***the square root of x***. Similarly, $\sqrt[3]{x}$ is called ***the cube root of x*** and denotes that number which when cubed gives x. In general, $\sqrt[n]{x}$ is called ***the nth root of x***, and denotes that number which when raised to the nth power gives x. The letter n as thus used in $\sqrt[n]{x}$ is called the ***index*** of the root.

Whenever one or more letters are combined in such a way as to require any of the processes just described, the result is called an ***expression***.

Thus $2x+3y$, $ax-bxy$, $6mn-3\sqrt{m}+2\sqrt{n}$, and $2x+y^2-x^3+xyz$ are expressions.

An expression is read from left to right in the order in which the indicated processes occur. Indicated multiplications and divisions are to be carried out, in general, before the indicated additions and subtractions.

Thus $2x+y^2-x^3+xyz$ is read "Two x plus y square minus x cube plus xyz."

EXERCISES

Read each of the following expressions:

1. $2x^2$.
2. x^3y^2.
3. $a^2-2ab+b^2$.
4. m^3-n^3.
5. $3\sqrt{x}+5\sqrt[3]{y}$.
6. $\dfrac{x}{y}+\dfrac{z^2}{w}-\dfrac{xy}{2}$.
7. $\sqrt{x+y}$.
8. $\sqrt{x}+\sqrt{y}$.
9. $\dfrac{3x+4y}{m-n}$.
10. $\dfrac{ax^3+bx^2+cx+d}{m^3-n^3}$.
11. $\sqrt[3]{\dfrac{x+y}{x-y}}$.
12. $\sqrt[n]{a^m+b^r}$.

13. Express each of the following ideas in letters.

(*a*) The sum of the squares of x and y.

(*b*) The difference between m cube and n cube.

(*c*) Three times the product of mn diminished by twice the quotient of x divided by the square root of y.

14. The fact that the area of any rectangle is equal to the product of its two dimensions (length and breadth) is expressed by the formula $A=ab$. Express similarly in words the meaning of each of the following familiar formulas:

(*a*) $A=\frac{1}{2}bh$. (Formula for the area of a triangle.)
(*b*) $A=\pi r^2$. (Formula for the area of a circle.)
(*c*) $C=2\pi r$. (Formula for the circumference of a circle.)
(*d*) $h^2=a^2+b^2$. (Theorem of Pythagoras concerning any right triangle.)
(*e*) $V=\frac{4}{3}\pi r^3$. (Formula for the volume of a sphere.)
(*f*) $A=4\pi r^2$. (Formula for the area of a sphere.)

4. Evaluation of Expressions. Whenever the values of the letters in an expression are given, the expression itself takes on a definite value. To obtain this value, we must work out the values of the separate parts of the expression and then combine them as indicated.

Example. Find the value of the expression

$$\frac{a^2+2bc-c^3}{a+b+c}$$

when $a=1$, $b=-2$, and $c=3$.

Solution. Giving a, b, and c their assigned values, the expression becomes

$$\frac{1^2+2\cdot(-2)\cdot 3-3^3}{1+(-2)+3}=\frac{1+(-12)+(-27)}{+2}=\frac{-38}{+2}=-19. \quad \textit{Ans.}$$

In evaluating expressions, it is useful to remember the following general facts, which result from (*d*) of § 2:

(*a*) The sign of the product of an *even* number of negative factors is positive.

Ex. $(-2)\cdot(-3)\cdot(-1)\cdot(-4)=+24$.

(*b*) The sign of the product of an *odd* number of negative factors is negative.

Ex. $(-2)\cdot(-3)\cdot(-1)=-6.$

(*c*) A negative number raised to an *even* power gives a positive result, but if raised to an *odd* power gives a negative result.

Ex. $(-2)^4=+16$, but $(-2)^3=-8.$

(*d*) An odd root of a negative number is negative.

Ex. $\sqrt[3]{-27}=-3$; $\sqrt[5]{-32}=-2$; $\sqrt[3]{-\frac{1}{8}}=-\frac{1}{2}.$

EXERCISES

Evaluate each of the following expressions for the indicated values of the letters.

1. $a^2+2ab+b^2$, when $a=1$, $b=-1$.

2. $4x^2y+4xy^2+xyz$, when $x=-1$, $y=2$, $z=-3$

3. $\dfrac{mn^2+n^3}{2m-n}$, when $m=2$, $n=6$.

4. $2\sqrt{2x}-3\sqrt[3]{9y}$, when $x=18$, $y=-3$.

5. $\dfrac{x}{2\sqrt{y}}-\dfrac{y}{3\sqrt{x}}$, when $x=9$, $y=16$.

6. $\sqrt{6xy+x^3+8z^2}$, when $x=-1$, $y=-3$, and $z=1$.

7. $\sqrt[3]{\dfrac{m^2+n^2}{x-y}}$, when $m=2$, $n=-2$, $x=5$, $y=6$.

8. By means of the formulas in Ex. 14, p. 6, find

(*a*) The area of the circle whose radius is 3 feet.

[Hint. Take $\pi=3\frac{1}{7}$.]

(*b*) The circumference of the circle whose radius is $1\frac{1}{2}$ feet.

(*c*) The volume of the sphere whose radius is 5 inches.

(*d*) The area of the sphere whose diameter is 1 yard.

5. Definitions. A ***monomial*** (or ***term***) is an expression not separated into parts by the signs + or −, as $5\,x^2y$.

A ***binomial*** is an expression having two terms, as $3\,x^2-4\,yx$.

A ***trinomial*** is an expression having three terms, as $3\,x^2-4\,xy+z^3$.

Any expression containing only powers of one or more letters is called a ***polynomial***.

A polynomial is said to be *arranged in descending powers* of one of its letters if the term containing the highest power of that letter is placed first, the term containing the next lower power is placed second, and so on.

Thus, if we arrange $\frac{1}{2}\,x^3+\frac{1}{4}\,x^5-1+x-3\,x^4$ in descending powers of x, it becomes $\frac{1}{4}\,x^5-3\,x^4+\frac{1}{2}\,x^3+x-1$.

Similarly, a polynomial is said to be *arranged in ascending powers* of one of its letters if the term containing the lowest power of that letter is placed first, the term containing the next higher power is placed second, and so on.

Thus, if we arrange $\frac{1}{2}\,x^3+\frac{1}{4}\,x^5-1+x-3\,x^4$ in ascending powers of x, it becomes $-1+x+\frac{1}{2}\,x^3-3\,x^4+\frac{1}{4}\,x^5$.

Whenever a term is broken up into two factors, either factor is known as the ***coefficient*** of the other one. Usually the word is used to designate the factor written first.

Thus, in $4\,xy$, 4 is the coefficient of xy; in ax, a is the coefficient of x, etc.

A ***common factor*** of two or more terms is a factor that occurs in each of them.

Thus $5\,x$, ax, x^2, and x^3 have the common factor x.

Whenever two or more terms have a common literal factor, they are said to be ***like terms*** with respect to that factor.

Thus $5\,x$, ax, x, and $-2\,x$ are like terms with respect to x; and $2\,a(x-y)$ and $3\,b(x-y)$ are like terms with respect to $x-y$.

6. Addition and Subtraction of Expressions. The following rules will be recalled from the *First Course*, pp. 49–57.

(*a*) *To add like terms, add the coefficients for a new coefficient and multiply the result by the common factor.*

Thus $3x+5x-4x=(3+5-4)x=4x$.
Similarly, $m(x-y)+n(x-y)=(m+n)(x-y)$.

(*b*) *To add polynomials, write like terms in the same column, find the sum of the terms in each column, and connect the results with the proper signs.*

Thus, in adding $3a+4b+2c$, $5a+3b-2c$, and $7a-9b-5c$, the work appears as follows:

$$\begin{array}{r} 3a+4b+2c \\ 5a+3b-2c \\ 7a-9b-5c \\ \hline 15a-2b-5c. \end{array} \quad \textit{Ans.}$$

(*c*) *To subtract a term from another like term, change the sign of the subtrahend and add the result to the minuend.*

Thus $8x^2y-(-3x^2y)=8x^2y+3x^2y=11x^2y$.

(*d*) *To subtract one polynomial from another, change all signs in the subtrahend and add the result to the minuend.*

Thus, in subtracting $4mn-2nr+3p$ from $5mn-4nr-4p$, what we have to do is to *add* $-4mn+2nr-3p$ to $5mn-4nr-4p$. Adding these (see the preceding rule for addition of polynomials) gives $mn-2nr-7p$. *Ans.*

NOTE. If two or more expressions can be arranged according to the descending powers of some letter (§ 5), it is usually best to do so before attempting to add, subtract, or perform other operations upon them.

7. Parenthesis ( ), Bracket [ ], Brace { }, and Vinculum $\overline{\quad}$. These are symbols for grouping terms that are to be taken as one single number or expression.

Thus $4x-(x+3y-z)$ means that $x+3y-z$ *as a whole* is to be subtracted from $4x$.

The following rules will be recalled:

(*a*) *A parenthesis preceded by the sign* $+$ (*either expressed or understood*) *may always be removed without any other change.*

(*b*) *A parenthesis preceded by the sign* $-$ *may be removed provided the sign of each term in the parenthesis be first changed.*

Thus $2a+3b+(x-3y+z)=2a+3b+x-3y+z$

but $2a+3b-(x-3y+z)=2a+3b-x+3y-z$.

EXERCISES

1. State the common factors in each of the following expressions.

(*a*) $-4x$, $-5x$, $6x$.
(*b*) rs, $3rs$, $-10r^2s$.
(*c*) $a(x+1)^2$, $b(x+1)$, $c(x+1)^3$.
(*d*) $2mn(a+b)$, $4m^2n(a-b)$, $8mn^2$.

2. Add $7a+6b-3c$ and $4a-7b+4c$.

3. Add $2x+3y-2xy$, $7xy-4x-9y$, and $7x-5xy-4y$.

4. Add $x-8-7x^2+15x^3$, $4+14x^3-11x-x^2$, and $5x^2-9x^3+10x-12$.

[HINT. See Note, § 6.]

5. Add $4(m+n)-3(q-r)$ and $4(m+n)+6(q-r)$.

6. Add $10(a+b)-11(b+c)$, $3(a+b)-5(c+d)$, and $3(b+c)-4(c+d)$.

7. Add $2mx+3nx-4qx$, $nx+2qx-ry$, and $py-qz+3w$.

8. Subtract $3x-2y+z$ from $5x-y+z$.

9. Subtract $4x^3-8-13x^2+15x$ from $6x^2+19x^3-4+12x$.

[HINT. See Note, § 6.]

10. From $13a+5b-4c$ subtract $8a+9b+10c$.

11. From $2a+3c+d$ subtract $a-b+c$.

12. From $3x^2+7x+10$ subtract $-x^2-x-6$.

13. Subtract $1-a+a^2-a^3$ from $1-a^3$.

14. From the sum of $x^2-4xy+y^2$ and $6x^2-2xy+3y^2$ subtract $3x^2-5xy+7y^2$. Do it all in one operation if you can.

15. From the sum of $2s+8t-4w$ and $3s-6t+2w$ take the sum of $8s+9t+6w$ and $4s-7t-4w$.

Remove the parentheses and combine terms in each of the following expressions.

16. $a-(2a+4)-(5a+10)$.

17. $6x+(5x-\{2x+1\})$.

[HINT. Remove the innermost group sign first.]

18. $x-\{x-(x-3x)\}$.

19. $20z-[(2z+7w)-(3z+5w)]$.

Find the values of the following when $a=4$, $b=3$, $c=-2$, and $d=-1$.

20. $10c^2-(3a+b+d)$.

[HINT. Simplify the expression as far as possible before giving to a, b, c, and d their special values.]

21. $3d-\{a-(c-b)\}$.

22. $a+\{c-(3d-b)\}+3\{(a-c)-7(b+d)\}$.

23. $\dfrac{2c-(a^2+b^2)}{2d-(a^2+c^2)}$.

24. $\sqrt{4\{4a^2-2(b^2+c^2+d^2)\}}$.

8. Multiplication The following formulas and rules will be recalled from the *First Course*.

Formula I. $\qquad x^m x^n = x^{m+n}$.

Thus $2^2 \cdot 2^3 = 2^5$; $x^2 \cdot x^3 = x^5$; $(2a)^3 \cdot (2a)^4 = (2a)^7$; $(a+b)^5 \cdot (a+b)^8 = (a+b)^{13}$; etc.

Formula I leads to the following rule.

(*a*) *To multiply one monomial by another, multiply the coefficients for the new coefficient and multiply the letters together, observing Formula I.*

Thus, in multiplying $-4m^2n^3$ by $2m^2n^2$ the new coefficient is $(-4)\times 2$, or -8, and the product of the letters is $m^2n^3m^2n^2$, or $m^2m^2n^3n^2$, which reduces by Formula I to m^4n^5. The answer, therefore, is $-8m^4n^5$.

Formula II. $(xy)^m = x^m y^m.$

Thus $(2 \cdot 3)^2 = 2^2 \cdot 3^2$; $(xy)^3 = x^3y^3$; $(3\,y)^3 = 3^3 \cdot y^3 = 27\,y^3$; $(2\,mn)^2 = (2\,m \cdot n)^2 = (2\,m)^2 \cdot n^2 = 2^2 \cdot m^2 \cdot n^2 = 4\,m^2n^2$; etc.

Formula III. $a(b+c) = ab+ac.$

Thus $2(3+4) = 2 \cdot 3 + 2 \cdot 4 = 6+8 = 14$; $x(y^2+z^3) = xy^2+xz^3$; $m^2(mn^2 - m^2n) = m^3n^2 - m^4n$; etc.

Formula III leads to the following rules.

(*b*) *To multiply a polynomial by a monomial multiply each term of the polynomial separately and combine the results.*

Thus the process of multiplying the polynomial $m - n + mn$ by the monomial mn is as follows:

$$m - n + mn$$

$$m^2n - mn^2 + m^2n^2. \quad \textit{Ans.}$$

(*c*) *To multiply one polynomial by another, multiply the multiplicand by each term of the multiplier and combine results.*

Thus the process of multiplying $x-y+3\,z$ by $2\,x+3\,y-z$ is as follows:

$$\begin{array}{lrrrrrr}
 & x & -\,y & +3\,z & & & \\
 & 2\,x & +3\,y & -\,z & & & \\
\hline
\text{Multiplying by } 2\,x, & 2\,x^2 & -2\,xy & +6\,xz & & & \\
\text{Multiplying by } 3\,y, & & 3\,xy & & -3\,y^2 & +\ 9\,yz & \\
\text{Multiplying by } -z, & & & -\ \ xz & & +\ \ \ yz & -3\,z^2 \\
\hline
\text{Combining results,} & 2\,x^2 & +\ \ xy & +5\,xz & -3\,y^2 & +10\,yz & -3\,z^2.
\end{array} \quad \textit{Ans.}$$

EXERCISES

Find the product in each of the following indicated multiplications.

1. $10\,a^5 \times 6\,a^2.$
2. $-2\,ab \times 3\,ab.$
3. $-4\,m^2n^3 \times 2\,m^2n.$
4. $3\,a^2b^2c^3 \times -2\,a^3b^2c^2.$
5. $4\,xyz^2 \times -8\,x^2yz.$
6. $(-4\,a^2bc^2) \times (2\,ab) \times (-3\,ac).$

[Hint to Ex. 6. Multiply the first two expressions together, then multiply this product by the last expression.]

Simplify each of the following expressions.

7. $(3\,x)^2$. **8.** $(3\,ab)^2$.

[HINT TO EX. 8. See fourth illustration under **Formula II.**]

9. $(2\,mn)^3$. **10.** $(8\,abc)^2$.

[HINT TO EX. 10. $8\,abc$ may be written $8\,a \cdot bc$.]

11. $(-2\,mn)^4$. **12.** $(-2\,x^2y^2)^3$.

13. Show that if the side of one square is twice that of another, its area is four times as great.

[HINT. Let a = a side of the small square. Then, a side of the large square $= 2\,a$, and the area $= (2\,a)^2$. Now apply Formula II.]

14. Show that if the edge of one cube is twice that of another, the volume is eight times as great.

15. Compare the areas of two circles, one of which has a radius three times as great as the other. (See Ex. 14 (*b*), p. 6.)

16. Compare the volumes of two spheres, one of which has a radius twice as great as the other. (See Ex. 14 (*e*), p. 6.)

Find the product in each of the following multiplications.

17. $(10\,a^3b + 7\,ab^4) \times -2\,a^2b$.

18. $(2\,x^2 - 3\,xy + 5\,y^2) \times -2\,xy$.

19. $(a^2 - 10\,ab + 15\,b^2) \times 4\,a^2b^2$.

20. $(a^2 - 2\,ab + b^2) \times (a - b)$.

21. $(2\,x + 7) \times (3\,x + 5)$.

22. $(4\,a^2 - 10\,b + 1) \times (2\,a^2 - b + 2)$.

23. Simplify the expression $(2\,x + y)^2$.

SOLUTION. $(2\,x + y)^2 = (2\,x + y) \cdot (2\,x + y)$. Multiplying gives $4\,x^2 + 4\,xy + y^2$. *Ans.*

24. If the side of a square is represented by $3\,x - 2$, what represents its area?

[HINT. Simplify your answer as in Ex. 23.]

25. If the dimensions of a rectangle are represented by $x + 2$ and $x - 1$, what represents its area?

26. What represents the area of the triangle whose base is $2x+3$ and whose altitude is $x-5$?

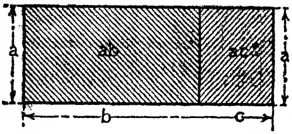


FIG. 2.

27. What represents the volume of the sphere whose radius is $2x-1$?

28. Explain how the figure to the left illustrates the geometric meaning of Formula III.

9. Division. The following formula and rules will be recalled from the *First Course.*

Formula IV. $$\frac{x^m}{x^n}=x^{(m-n)}.$$

Thus $\frac{2^5}{2^3}=2^{5-3}=2^2$; $\frac{x^7}{x^3}=x^4$; $\frac{(2a)^6}{(2a)^2}=(2a)^4=2^4a^4=16a^4$;

$$\frac{(a+b)^5}{(a+b)^3}=(a+b)^2;\ \text{etc.}$$

Formula IV leads to the following rules.

(*a*) *To divide one monomial by another, divide the coefficients for the new coefficient, observing the law of signs for division* (§ 2 (*e*)), *and divide the literal factors, observing Formula IV.*

Thus, in dividing $28a^3b^2$ by $-4ab$ the process is carried out as follows:

$$\begin{array}{r|l} -4ab & 28a^3b^2 \\ \hline & -7a^2b. \quad \textit{Ans.} \end{array}$$

Here the division of 28 by -4 gives the new coefficient, -7, then the division of a^3 by a gives a^2 (by Formula IV), and finally the division of b^2 by b gives b.

(*b*) *To divide a polynomial by a monomial, divide each term of the polynomial separately, and combine results.*

Thus the division of $8x^2y-4x^3y^2+2xy$ by $2xy$ is carried out as below:

$$\begin{array}{r|l} 2xy & 8x^2y-4x^3y^2+2xy \\ \hline & 4x \quad -2x^2y \quad +1. \quad \textit{Ans.} \end{array}$$

(*c*) *To divide one polynomial by another:*

1. *Arrange the dividend and divisor in the descending (or ascending) powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

3. *Multiply the whole divisor by the first term of the quotient; write the product under the dividend and subtract it from the dividend.*

4. *Consider the remainder as a new dividend, and repeat steps* 1, 2, *and* 3, *continuing in the same manner thereafter.*

Thus the division of $17x+20+3x^2$ by $4+x$ is carried out as follows:

$$\begin{array}{r|l} 3x^2+17x+20 & \underline{x+4} \\ \underline{3x^2+12x} & 3x+5 \quad \textit{Quotient. Ans.} \\ 5x+20 & \\ \underline{5x+20} & \\ 0 & \end{array}$$

In this example, a new dividend is finally obtained which is equal to 0. Whenever this happens, the division is said to be *exact*. In cases where the division does not come out in this way, there is a *remainder*, as illustrated by the following example:

$$\begin{array}{r|l} 3x^2+17x+20 & \underline{x+3} \\ \underline{3x^2+9x} & 3x+8 \quad \textit{Quotient. Ans.} \\ 8x+20 & \\ \underline{8x+24} & \\ -4 & \textit{Remainder.} \end{array}$$

EXERCISES

1. How do you test (or check) to see whether an answer in division is correct? Is the same method used for this in both arithmetic and algebra? Check the correctness of the results obtained for the last two examples in § 9.

Perform the following divisions, and check your answer.

2. $a^6 \div a^2$. **3.** $(3\,g)^9 \div (3\,g)^2$. **4.** $(3\,ab)^8 \div (3\,ab)^4$.

5. $(\frac{1}{2}\,p^2q)^7 \div (\frac{1}{2}\,p^2q)^3$. **8.** $\frac{4}{3}\,\pi r^3 \div 2\,\pi r$.

6. $-16\,x^2y^3z^2 \div 4\,xy^2z$. **9.** $3\,ab(a+b)^2 \div [-2(a+b)]$.

7. $4\,a^4b^2c^3 \div 20\,a^2b^2c$. **10.** $(9\,m^3np + 18\,mn^3p) \div 3\,mn$.

11. $(6\,x^2yz + 12\,xy^2z - 24\,xyz^2) \div (-3\,xyz)$.

In each of the following divisions, find the quotient, also the remainder if there is one. Check your answer for each.

12. $(3\,x^2 - 2\,x - 1) \div (x-1)$. **13.** $(15\,x^2 + x - 2) \div (3\,x - 1)$.

14. $(4\,y^3 + 2\,y^2 - 1) \div (2\,y - 1)$.

[Hint. Write the dividend in the form $4\,y^3 + 2\,y^2 + 0\,y - 1$.]

15. $(6\,x^3 - 7\,x^2 + 1) \div (2\,x - 1)$.

16. $(x^3 + x^2 - x + 2) \div (x^2 - x + 1)$.

17. $(2\,x^2 + 3\,x + 1) \div (x+2)$.

18. $(x^4 - 3\,x^3 + x^2 + 2\,x - 1) \div (x^2 - x - 2)$.

19. $(a^3 - 2\,a^2b + 2\,ab^2 - b^3) \div (a-b)$.

20. $(x^4 - y^4) \div (x+y)$.

MISCELLANEOUS EXERCISES

1. Add $4\,x^3 - 2\,x^2 - 7\,x + 1$, $x^3 + 3\,x^2 + 5\,x - 6$, $4\,x^2 - 8\,x^3 + 2 - 6\,x$, $2\,x^3 - 2\,x^2 + 8\,x + 4$, and $-2\,x + 1 + 2\,x^3 - 3\,x^2$.

2. Add $3(a+b) + 6(b+c)$, $5(a+b) - 10(b+c)$, $2(a+b) + (b+c)$, $3(b+c) - (a+b)$, $2(b+c) - 10(a+b)$, and $3(a+b) - 3(b+c)$.

3. From the sum of $1+x$ and $1-x^2$ subtract $1 - x + x^2 - x^3$.

4. Simplify the expression

$$ab - \{5 + x - (b + c - ab + x)\} + [x - (b - c - 7)].$$

5. When $x=3$, $m=6$, $n=2$ find the value of the expression

$$(m+n+x)^n - (m+n-x)^n - (m-n+x)^n(-m+n+x)^n.$$

6. Simplify the expression

$$y^3 - [2\,x^3 - xy(x-y) - y^3] + 2(x-y)(x^2+xy+y^2).$$

7. Find the remainder in the following divisions

(a) $(a^4+1) \div (a-1)$.

(b) $[x^{3n-3} + y^{3n+3}] \div [x^{n-1} + y^{n+1}]$.

CHAPTER II

SPECIAL PRODUCTS AND FACTORING

10. Special Products. Certain products occur so frequently in algebra that it is desirable to study them with especial care and to remember their forms. In this connection, the following formulas will be recalled.

Formula V. $\quad (x-y)(x+y)=x^2-y^2.$

Thus $(x-8)(x+8)=x^2-8^2=x^2-64$;

$(9x-3y)(9x+3y)=(9x)^2-(3y)^2=81x^2-9y^2.$

Formula VI. $\quad (x+y)^2=x^2+2xy+y^2.$

Thus $\quad (r+6)^2=r^2+2(r\cdot 6)+6^2=r^2+12r+36$;

$$(2a+3b)^2=(2a)^2+2(2a)(3b)+(3b)^2=4a^2+12ab+9b^2.$$

Formula VII. $\quad (x-y)^2=x^2-2xy+y^2.$

Thus $\quad (r-6)^2=r^2-2(r\cdot 6)+6^2=r^2-12r+36$;

$$(3x-2y)^2=(3x)^2-2(3x)(2y)+(2y)^2=9x^2-12xy+4y^2.$$

Formula VIII. $\quad (x+m)(x+n)=x^2+(m+n)x+mn,$ *m and n being any (positive or negative) numbers.*

Thus $(x+4)(x+3)=x^2+(4+3)x+4\cdot 3=x^2+7x+12$;

$$(ab-6)(ab+2)=(ab)^2+(-6+2)ab+(-6)\cdot 2=a^2b^2-4ab-12.$$

ORAL EXERCISES

State under which formula each of the following products comes, and read off the answer by inspection.

1. $(x-3)(x+3)$.
2. $(r-5)(r+5)$.
3. $(6+a)(6-a)$.
4. $(x-4)(x+4)$.
5. $(5x-2)(5x+2)$.
6. $(1-3x^2)(1+3x^2)$.
7. $(xy-12)(xy+12)$.
8. $\left(\frac{a}{2}+\frac{b}{3}\right)\left(\frac{a}{2}-\frac{b}{3}\right)$.
9. $(x+8)^2$.
10. $(x-8)^2$.
11. $(2x+1)^2$.
12. $(3x-4)^2$.
13. $(a+2b)^2$.
14. $(4xy-1)^2$.
15. $(2ab-3cd)^2$.
16. $(x+\frac{1}{2})^2$.
17. $(3x-\frac{2}{3})^2$.
18. $(2m+\frac{1}{3}n)^2$.
19. $(x+3)(x-4)$.
20. $(a+6)(a-8)$.
21. $(x-12)(x-2)$.
22. $(ab+2)(ab+6)$.
23. $(a^2+4)(a^2-6)$.
24. $(4x+3)(4x-2)$.
25. $(3a-5y)(3a+7y)$.
26. $\{(a+b)+5\}\{(a+b)-5\}$.
27. $\{(m+n)-2p\}\{(m+n)+2p\}$.
28. $\{8-(r+s)\}\{8+(r+s)\}$.
29. $\{(a+b)-(c+d)\}\{(a+b)+(c+d)\}$.
30. $\{(m+n)-4\}\{(m+n)-5\}$.

WRITTEN EXERCISES

1. Show how Formulas VI and VII can be obtained as special cases of Formula VIII.

2. Show how the following three figures illustrate respectively the geometric meanings of Formulas V, VI, and VII.

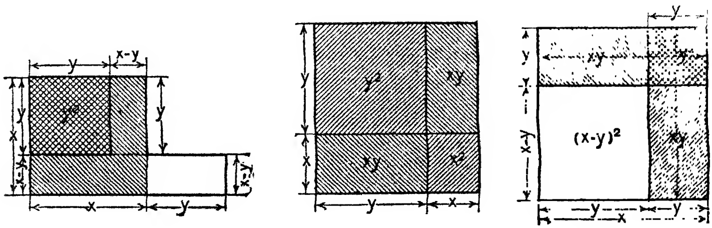


FIG. 3.

By use of Formulas V, VI, and VII write down (without multiplying out) the simplest forms for the following products.

3. $(a^2+a-1)(a^2-a+1)$.

[HINT. Write first as $\{a^2+(a-1)\}\{a^2-(a-1)\}$, then apply Formula V, afterwards simplifying your answer by Formula VII. The final result is a^4-a^2+2a-1.]

4. $(a^2+ab+b^2)(a^2-ab+b^2)$.

5. $(x^2+x-2)(x^2-x-2)$.

6. $(a-b+m+n)(a-b-m-n)$.

7. $\{(x+y)-4\}^2$.

8. $[7+(m-n)]^2$.

9. $[(x+y)-(m+n)]^2$.

10. $(a+b+c)^2$.

[HINT. Write as $[(a+b)+c]^2$ and apply Formula VI twice.]

11. The rule for finding the square of *any* polynomial is as follows. *The square of any polynomial is equal to the sum of the squares of its terms plus twice the product of each term by each term that follows it.* For example,

$$(a+b+c+d)^2 =$$
$$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Show that Formulas VI and VII conform to this general rule; also that your answer for Ex. 10 does so.

By means of the general rule in Ex. 11, write out the values of each of the following expressions.

12. $(a+b-c)^2$.

13. $(a-b-c)^2$.

14. $(2x+y-z)^2$.

15. $(2x+2y-z+3w)^2$.

11. Type Forms of Factoring. Factoring is the reverse of multiplication in the sense that in multiplication certain factors are given and we are asked to find their product, while in factoring a certain product is given and we are asked to find its factors, that is, to find expressions which multiplied together produce it. The following four types of expressions are to be especially noted, as they can always be readily factored.

(*a*) *Expressions whose terms each contains a common factor.*

Thus $mx+my+mz=m(x+y+z)$; (Formula III, p. 12.)

$$a^2x+ax^2+a^2x^2=ax(a+x+ax);$$
$$ax-ay+bx-by=a(x-y)+b(x-y)=(x-y)(a+b);$$
$$2x-y+4x^2-2xy=(2x-y)+2x(2x-y)=(2x-y)(1+2x).$$

Note that in all these examples the given expression has finally been exhibited as a *product*. This is essential to every example in factoring.

(*b*) *Expressions which can be regarded as the difference of two squares.*

Thus $25x^2-y^2=(5x-y)(5x+y)$; (Formula V, p. 17.)

$$a^2b^2-c^2d^2=(ab-cd)(ab+cd);$$
$$\tfrac{1}{4}x^2-\tfrac{9}{16}y^4=(\tfrac{1}{2}x-\tfrac{3}{4}y^2)(\tfrac{1}{2}x+\tfrac{3}{4}y^2);$$
$$(a+b)^2-c^2=[(a+b)-c][(a+b)+c]=(a+b-c)(a+b+c).$$

(*c*) *Trinomials of the form* $\boldsymbol{x^2+px+q}$, *where* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *have such values that we can readily find two numbers whose* **sum** *is* $\boldsymbol{p}$ *and whose* **product** *is* $\boldsymbol{q}$.

Thus, in factoring $x^2+7x+12$, we need only inquire whether we can find two numbers whose sum is 7 and whose product is 12. The numbers 3 and 4 are seen (by inspection) to do this. Hence we know by Formula VIII that we may write

$$x^2+7x+12=(x+3)(x+4). \quad \textit{Ans.}$$

Similarly, $x^2-x-12=(x-4)(x+3)$; Why?

$$x^2-5xy-36y^2=(x-9y)(x+4y);$$
$$a^2b^2-21ab-72=(ab-24)(ab+3).$$

(*d*) *Trinomials of the form* $\boldsymbol{ax^2+bx+c}$ *which are perfect squares*, that is such that the coefficients a and c are perfect squares while the other coefficient, namely b, is equal in absolute value (§ 1) to twice the product of the square roots of a and c.

Thus $9x^2+12x+4$ is a perfect square because $12=2\cdot\sqrt{9}\cdot\sqrt{4}$. Hence, by Formula VI, we have

$$9x^2+12x+4=(3x+2)^2=(3x+2)(3x+2). \quad \textit{Ans.}$$

Similarly, $x^2-14x+49$ is a perfect square, because

$$14=2\cdot\sqrt{1}\cdot\sqrt{49}.$$

Thus we have, using Formula VII,

$$x^2-14x+49=(x-7)^2=(x-7)(x-7).$$

For a like reason, we see that $x^2+14x+49=(x+7)(x+7)$.

The following are two other examples that can be brought under this case.

$$a^2b^2-2ab+4=(ab-2)^2=(ab-2)(ab-2).$$

$$4(a+b)^2+4(a+b)+1=[2(a+b)+1]^2=(2a+2b+1)(2a+2b+1).$$

We could not, however, factor x^2+3x+4 or $4x^2-4x-1$ by this method. Why?

EXERCISES

Factor each of the following expressions:

1. (a) x^2+2x. (c) $8a^2+24a$.
   (b) x^2y+xy^2. (d) $ab^2+b^3-b^2c$.
   (e) $25c^2dx^3+35c^3d^2x^4-55c^2d^2x^5$.
   (f) $m(3x-1)-n(3x-1)$.
   (g) $m(a+b-c)+n(a+b-c)-q(a+b-c)$.
   (h) $pq-px-rq+rx$.
   (i) $y^2-4y+xy-4x$. (j) $3x^3-15x+10y-2x^2y$.

2. (a) $81-x^2$. (f) $9b^2-(a-x)^2$.
   (b) $a^2-b^2c^2$. (g) $49a^2-(5a-4b)^2$.
   (c) $144x^2-4$. (h) $(2x+5)^2-(5x-3)^2$.
   (d) $\frac{1}{4}x^2y^2-36$. (i) $(x+x^2)^2-(2x+2)^2$.
   (e) $\frac{a^2}{b^2}-\frac{x^2}{y^2}$. (j) $(a+b+c)^2-(a-b-c)^2$.

3. (a) x^2+6x+8. (f) $x^2+xy-56y^2$.
   (b) x^2-6x+8. (g) $12+7a+a^2$.
   (c) y^2+y-42. (h) $y^2-3ny-28n^2$.
   (d) $x^2-13x-48$. (i) $a^2x^2+ax-12$.
   (e) $x^2-x-110$. (j) $x^4+19cx^2+90c^2$.

4. Test each of the following expressions to see whether it is a trinomial square, and if so, factor it.

(*a*) $x^2-8x+16$.
(*b*) $x^2-12x+36$.
(*c*) $4x^2+6x+1$.
(*d*) $81x^2+18x+1$.
(*e*) $81-72r+16r^2$.
(*f*) $16y^2-24y+9$.
(*g*) $4x^2y^2-20xy+25$.
(*h*) $x^2+2x(x-y)+(x-y)^2$.
(*i*) $16-24(a-b)+9(a-b)^2$.
(*j*) $(a+b)^2-2(a+b)(b+c)+(b+c)^2$.

5. The figure shows a square of side a within which lies (in any manner) a smaller square of side b. Prove that the area between the two (shaded in the figure) is equal to $(a+b)(a-b)$.

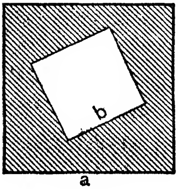


FIG. 4.

6. The result in Ex. 5 furnishes a rule for determining quickly the area between any two squares when the one lies within the other. State the rule.

7. By means of Exs. 5 and 6 answer the following: What is the area of pavement in the street surrounding a city block one half mile on a side, the street being 4 rods wide? (1 mile = 320 rods.)

8. Show that the area between a circle of radius R and a smaller circle of radius r lying within it is equal to $\pi(R+r)(R-r)$. Does it make any difference where the smaller circle lies so long as it is within the large one?

9. Show that if a and b are the sides of a right triangle whose hypotenuse is h, we shall have $a^2=(h+b)(h-b)$; also $b^2=(h+a)(h-a)$.

10. Formula V is frequently used to find the square of a number quickly by mental arithmetic. Suppose, for example, that we wish to know the value of 16^2. We first take 6 away from the number, leaving 10, then we add 6 to it, giving 22.

We multiply the 10 and the 22 thus obtained (as is easily done mentally), giving 220. Now all we have to do is to add 6^2, or 36, to the 220 to obtain the desired value of 16^2, giving 256 as the answer. The reason for these steps appears below.

$$(16-6)(16+6)=16^2-6^2, \quad \text{(Formula V.)}$$

whence $\quad (16-6)(16+6)+6^2=16^2.$

Find (mentally) in this way the value of each of the following expressions.

(*a*) 15^2. (*c*) 17^2. (*e*) 31^2.

[HINT. Subtract 5 first, then add 5.]

(*b*) 14^2. (*d*) 22^2. (*f*) 45^2.

* **12. Other Type Forms.** Besides the type forms mentioned in § 11, the following may be noted:

(*e*) *Trinomials of the form* ax^2+bx+c *which are* **not** *perfect squares and hence do not fall under* (*d*) *of* § 11. There is no general rule in such cases, though we may frequently discover *by inspection* whether a given trinomial of this form is factorable readily, and if so, obtain its factors. This is best understood from an example.

EXAMPLE. Factor $15x^2-7x-2$.

SOLUTION. For $15x^2$, try $5x$ and $3x$; thus we begin by writing $(5x \qquad)(3x \qquad)$, where the open spaces are yet to be filled in. For -2 try 1 and 2 with unlike signs, arranging the signs so that the sum of the cross products shall give, as desired, the $-7x$ of the given expression; thus we now try $(5x+1)(3x-2)$. Here the middle term of the product (cross product) is readily found to be $-10x+3x$, or $-7x$, as desired. The only other possibility would be $(5x-1)(3x+2)$, but as the cross product term here becomes $10x-3x$, or $+7x$, this form cannot be the one we desire.

We have, therefore, $15x^2-7x-2=(5x+1)(3x-2)$. *Ans.*

(*f*) *The sum of two cubes.* This form is factorable in accordance with the following formula.

Formula IX. $x^3+y^3=(x+y)(x^2-xy+y^2)$.

Thus $x^3+8=x^3+2^3=(x+2)(x^2-2x+2^2)$
$=(x+2)(x^2-2x+4)$. *Ans.*

Likewise,

$$m^3n^3+64p^3=(mn)^3+(4p)^3=(mn+4p)[(mn)^2-(mn)(4p)+(4p)^2]$$
$$=(mn+4p)(m^2n^2-4mnp+16p^2). \quad Ans.$$

(*g*) *The difference of two cubes.* This form is factorable in accordance with the following formula.

Formula X. $x^3-y^3=(x-y)(x^2+xy+y^2)$.

Thus $x^3-27=x^3-3^3=(x-3)(x^2+3x+3^2)$
$=(x-3)(x+3x+9)$. *Ans.*

$$8x^3-125y^3=(2x)^3-(5y)^3=(2x-5y)[(2x)^2+(2x)(5y)+(5y)^2]$$
$$=(2x-5y)(4x^2+10xy+25y^2). \quad Ans.$$

13. Complete Factoring. Each of the exercises on p. 21 concerns but a *single* one of the type forms mentioned in § 11, but we often meet with problems in which two or more of the types are concerned at the same time. Thus, in factoring a^3b-ab^3, we first take out the common factor ab. This gives $a^3b-ab^3=ab(a^2-b^2)$. But a^2-b^2 is itself factorable, coming under type (*b*) of § 11. The final answer, therefore, is $ab(a-b)(a+b)$.

Other illustrations of this idea occur below. Note that the *final* answer in every case contains no factors which themselves can be still further broken up into other factors.

EXAMPLE 1. Factor completely x^2-y^2+x-y.

SOLUTION.

$$x^2-y^2+x-y=(x^2-y^2)+(x-y)=(x-y)(x+y)+(x-y) \quad \text{(See } (b), \S 11.)$$
$$=(x-y)(x+y+1). \quad Ans.$$

(See (*a*) § 11.)

EXAMPLE 2. Factor completely $a^4+3a^2b^2-4b^4$.

SOLUTION.

$$\begin{aligned} a^4+3a^2b^2-4b^4 &= (a^2+4b^2)(a^2-b^2) && \text{(See (c), § 11.)} \\ &= (a^2+4b^2)(a-b)(a+b). && \text{(See (b), § 11.)} \end{aligned}$$

EXERCISES

Factor *completely* each of the following expressions. Those preceded by the * involve the type forms mentioned in § 12.

1. x^3-x^2-x+1.

[HINT. Write in the form $x^2(x-1)-(x-1)$.]

2. $2x^3-8x^2y+8xy^2$.

[HINT. Write in the form $2x(x^2-4xy+4y^2)$.]

3. $x^2+3ax-3a-x$.

4. a^3+2a^2-4a-8.

5. x^4-13x^2+36.

6. $x^4+y^4-2x^2y^2$.

7. $x^4-(2x-1)^2$.

8. $1-a^2-b^2+2ab$.

[HINT. $1-a^2-b^2+2ab=1-(a-b)^2$.]

9. $m^2-4mn+4n^2-16$.

10. $2xy-x^2-y^2+1$.

11. $1+9c^2+6c$.

12. $(x^2-1)^2+(2x+3)(x-1)^2$.

13. $3x^3-3x+4x^4-4x^2$.

14. $1-a^2b^2-x^2y^2+2abxy$.

15. $(x^2-y^2)^2-(x^2-xy)^2$.

***16.** $x^3+y^3+x^2-y^2$.

***17.** $(x+1)^3-x^6$.

18. $(1-2x)^2-x^4$.

***19.** $6x^2+7x-3$.

***20.** $20x^2-6x-2$.

21. $8x^2-18xy-5y^2$.

22. x^4-16.

23. $11x^2+34x+3$.

24. $25s^4-25s^2t^2+4t^4$.

25. $9x^4-30x^2a+25a^2$.

26. $a^2+b^2+c^2-2ab+2ac-2bc$.

27. $(a-x^2)^2-(b-y^2)^2$.

***28.** x^3+y^3+x+y.

29. x^2-9y^2+x+3y.

30. $a^3-2a^2b+4ab^2-8b^3$.

31. $16x^4-y^2-6x^2y$.

32. $x^4-47x^2y^2+90y^4$.

CHAPTER III

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

14. Prime Factor. A number which has no factor except itself and unity is called in arithmetic a prime number. Such a number when used as a factor is called a ***prime factor.***

Thus the prime factors of 15 are 3 and 5.

The word *prime factor* is similarly used in algebra.

Thus we say that the prime factors of $3\,abc$ are 3, a, b, and c.

The prime factors of $18\,x^2y$ are 2, 3, 3, x, x, and y.

The prime factors of $a^2b(a^2-b^2)$ are a, a, b, $a-b$, and $a+b$. (See Formula V.)

15. Finding Common Factors. As soon as we have factored each of several expressions into its prime factors, we can readily pick out their common factors (§ 5).

Thus, in finding the common factors of abc, a^2b, ab^2, and $3\,ab$, we write

$abc = a \cdot b \cdot c,\qquad a^2b = a \cdot a \cdot b,\qquad ab^2 = a \cdot b \cdot b,\qquad 3\,ab = 3 \cdot a \cdot b.$

The common factors are, therefore, a and b, since these occur in *each* expression and they are the only factors thus appearing.

16. Highest Common Factor. The product of all the common prime factors of two or more expressions is called their ***highest common factor.*** It is called the highest because it contains *all* the common factors, and the usual abbreviation for it is ***H. C. F.***

EXAMPLE 1. Find the H. C. F. of $10x^2y^3$, $2xy^2$, and $18x^3y^2$.

SOLUTION. Resolving each into its prime factors,

$$10\,x^2y^3 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y,$$
$$2\,xy^2 = 2 \cdot x \cdot y \cdot y,$$
$$18\,x^3y^2 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y.$$

The factors *common* to the three expressions are thus seen to be 2, x, y, y.

The H. C. F. is, therefore, $2 \cdot x \cdot y \cdot y$, or $2\,xy^2$. *Ans.*

EXAMPLE 2. Find the H. C. F. of $3x^2+3x-18$, $6x^2+36x+54$, and $9x^2-81$.

SOLUTION.

$3x^2+3x-18=3(x^2+x-6)=3(x+3)(x-2)$. (See (*c*), § 11.)

$6x^2+36x+54=6(x^2+6x+9)=2\cdot 3(x+3)(x+3)$. (See (*d*), § 11.)

$9x^2-81=9(x^2-9)=3\cdot 3(x+3)(x-3)$. (See (*b*), § 11.)

The common factors being 3 and $(x+3)$, the H. C. F. is $3(x+3)$.) *Ans.*

In general, to find the H. C. F. of two or more expressions:

1. *Find the prime factors of each expression.*

2. *Pick out the different prime factors and give to each the lowest exponent to which it occurs in any of the expressions.*

3. *Form the product of all the factors found in step* 2.

NOTE. Since the H. C. F. of several expressions consists only of factors common to them all, it is always an exact divisor of each of the expressions. It is therefore called in arithmetic "the greatest common divisor" and is represented by G. C. D.

EXERCISES

Find the H. C. F. of each of the following groups.

1. 12, 18.
2. 16, 24, 36.
3. x^2y, xy^2.
4. a^2b, ab^2, a^2b^2.
5. $2x^2y$, $6x^3y$, $14x^3y^2z^4$.
6. a^2-b^2, $a^2-2ab+b^2$.
7. $a^2+7a+12$, a^2-9.
8. x^2-y^2, $(x-y)^2$, $x^2-3xy+2y^2$.
9. m^2+4m+4, $m^2-6m-16$.
10. $3y^2-363$, $y^2-7y-44$.
11. $2a^2+4a$, $4a^3+12a^2+8a$.
12. $^4-y^4$, $x^3+x^2y+xy^2+y^3$.
13. $3r^5+9r^4-3r^3$, $5r^2s^2+15rs^2-5s^2$, $7ar^2+21ar-7a$.

*14. a^3-b^3, a^2-b^2, $a-b$.

*15. x^3+1, x^2-x+1.

17. Common Multiple. In arithmetic a number which is exactly divisible by two or more given numbers is called a ***common multiple*** of them.

The word common multiple is similarly used in algebra.

Thus $4x^2y^2$ is a common multiple of x and y; it is also a common multiple of 4 and x. Similarly, a^2-b^2 is a common multiple of $a-b$ and $a+b$.

18. Lowest Common Multiple. The ***lowest common multiple*** of two or more numbers or expressions is that multiple of them which contains the fewest possible prime factors. Its abbreviation is ***L. C. M.***

The following examples illustrate what the L. C. M. means and how to obtain it.

EXAMPLE 1. Find the L. C. M. of $10\,a^2b$, $16\,a^2b^3$, and $20\,a^3b^4$.

SOLUTION. Separate each expression into its prime factors. Thus

$$10\,a^2b = 2\cdot 5\cdot a\cdot a\cdot b,$$
$$16\,a^2b^3 = 2\cdot 2\cdot 2\cdot 2\cdot a\cdot a\cdot b\cdot b\cdot b,$$
$$20\,a^3b^4 = 2\cdot 2\cdot 5\cdot a\cdot a\cdot a\cdot b\cdot b\cdot b\cdot b.$$

The L. C. M. is $2\cdot 2\cdot 2\cdot 2\cdot 5\cdot a\cdot a\cdot a\cdot b\cdot b\cdot b\cdot b$, or $80\,a^3b^4$, since this contains *all* the factors of each of the three expressions, and at the same time is made up of fewer factors than any other similar expression that can be found.

EXAMPLE 2. Find the L. C. M. of $x^4-10\,x^2+9$ and $x^2-4\,x+3$.

SOLUTION.

$$x^4-10\,x^2+9=(x^2-9)(x^2-1)=(x-3)(x+3)(x-1)(x+1),$$
$$x^2-4\,x+3=(x-3)(x-1).$$

Therefore the L. C. M. is $(x-3)(x+3)(x-1)(x+1)$.

In general, to find the L. C. M. of two or more expressions:

1. *Find the prime factors of each expression.*

2. *Pick out the different prime factors, taking each the greatest number of times it occurs in any one of the expressions.*

3. *Form the product of all the factors found in step 2.*

NOTE. From the manner in which the L. C. M. is formed, it must be exactly divisible by each of the given numbers, or expressions.

EXERCISES

Find the L. C. M. of each of the groups of expressions in the exercises on p. 27.

CHAPTER IV

FRACTIONS

19. Definitions. Any expression of the form a/b is called a ***fraction***. It means the number, or expression, which when multiplied by b gives a. The part above the line, or a, is called the ***numerator***, while the part below the line, or b, is called the ***denominator***. The numerator and denominator taken together are called the ***terms*** of the fraction.

20. Equivalent Fractions. It is often desirable to change the form of a fraction without changing its value. Such changes all depend upon the following principle.

The numerator and denominator of a fraction may be multiplied or divided by the same number, or expression, without changing the value of the fraction. Thus

$$\frac{3}{4}=\frac{3\cdot 2}{4\cdot 2}=\frac{6}{8};$$

$$\frac{4\,a}{5}=\frac{4\,a\cdot a}{5\cdot a}=\frac{4\,a^2}{5\,a};$$

$$\frac{a+b}{(a+b)^2}=\frac{\cancel{(a+b)}}{(a+b)\cancel{(a+b)}}=\frac{1}{a+b}.$$

21. Changes of Signs in Fractions. There are three signs to be considered in a fraction; the sign of the numerator, the sign of the denominator, and the sign of the fraction itself.

Thus in $+\frac{-3}{4}$, the three signs in the order just mentioned are $-$, $+$, $+$, while in $-\frac{5\,a}{-6\,b}$ they are $+$, $-$, $-$.

Since a fraction is merely an indicated division, the law of signs for division (§ 2 (*e*), p. 3) must hold at all times, so that we arrive at the following rule.

Any two of the three signs of a fraction may be changed without altering the value of the fraction. Thus

$$+\frac{+3}{+4}=+\frac{-3}{-4}=-\frac{-3}{+4}=-\frac{+3}{-4}.$$

Likewise

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

Care must be taken, however, in changing the sign of the numerator or denominator of a fraction when polynomials are present. For example, if the numerator is a po'ynomial, we can change the sign of the whole numerator only by changing the sign of *every* term in it. A similar statement applies when the denominator is a polynomial. Thus,

$$\frac{a+2b+c}{2a-3b-2c}=-\frac{-a-2b-c}{2a-3b-2c}=-\frac{a+2b+c}{-2a+3b+2c}.$$

Observe carefully the reason for every change of sign here.

22. Reduction of Fractions to Lowest Terms. A fraction is reduced to its lowest terms when its numerator and denominator have no common factor except 1.

To reduce a fraction to its lowest terms, factor numerator and denominator, then divide each by all their common factors.

EXAMPLE 1. $$\frac{25a^2b^3x}{35a^3bx}=\frac{\cancel{5}\cdot 5\cdot \cancel{a}\cdot \cancel{a}\cdot \cancel{b}\cdot b\cdot b\cdot \cancel{x}}{\cancel{5}\cdot 7\cdot \cancel{a}\cdot \cancel{a}\cdot a\cdot \cancel{b}\cdot \cancel{x}}=\frac{5b^2}{7a}.$$

EXAMPLE 2: $$\frac{a^2-11a+24}{a^2-a+6}=\frac{(a-8)\cancel{(a-3)}}{(a-2)\cancel{(a-3)}}=\frac{a-8}{a-2}.$$

23. Lowest Common Denominator. The lowest common denominator of two or more fractions is the lowest common multiple (§ 18) of their denominators.

To reduce several fractions to their lowest common denominator:

1. *Find the L. C. M. of the denominators.*

2. *For each fraction, divide this L. C. M. by the given denominator, and multiply both numerator and denominator by the quotient.*

Example. Reduce the following fractions to equivalent fractions having their lowest common denominator:

$$\frac{x+2}{x^2-9} \text{ and } \frac{x+5}{x^2+9x+18}.$$

Solution. Factoring the denominators, the fractions may be written

$$\frac{x+2}{(x+3)(x-3)} \text{ and } \frac{x+5}{(x+3)(x+6)}.$$

The L. C. M. of these denominators is $(x+3)(x-3)(x+6)$. In order to give the first fraction this L. C. M. as its denominator, multiply its numerator and denominator by $x+6$ (this being the L. C. M. divided by the denominator of the first fraction).

In order to give the second fraction this L. C. M. as its denominator, multiply its numerator and denominator by $x-3$ (this being the L. C. M. divided by the denominator of the second fraction). The desired forms are, therefore,

$$\frac{(x+2)(x+6)}{(x+3)(x-3)(x+6)} \text{ and } \frac{(x+5)(x-3)}{(x+3)(x-3)(x+6)}.$$

Observe that these fractions are respectively equivalent (§ 20) to those with which we started, but these have denominators that are *alike*, which was not the case with the original forms.

EXERCISES

Write each of the following fractions in three other ways without changing the value.

1. $-\dfrac{5}{-6}$. 2. $\dfrac{6}{1-x}$. 3. $\dfrac{a-b}{c-d}$. 4. $\dfrac{3x-4}{(2x-1)(x+3)}$

Reduce each of the following expressions to lowest terms.

5. $\dfrac{240}{320}$.

6. $\dfrac{10xy}{30x^2y^2}$.

7. $\dfrac{36xr^3}{72x^3r^3}$.

8. $\dfrac{-9a^2b^2c^2}{54a^2b^3c^2}$.

9. $\dfrac{x^2-y^2}{x^2-2xy+y^2}$.

10. $\dfrac{a^2b^2+ab}{(ab+1)^2}$.

11. $\dfrac{4x^2-y^2}{y-2x}$.

12. $\dfrac{s^2-6s+8}{s^2-5s+6}$.

13. $\dfrac{r^4-6r^2+5}{r^2-6r+5}$.

14. $\dfrac{9x^2-49y^2}{28xy^2-12x^2y}$.

Reduce all of the fractions in each of the following groups to the lowest common denominator.

15. $\frac{1}{3}, \frac{3}{4}, \frac{5}{6}$.

16. $\dfrac{a}{4}, \dfrac{b}{6}, \dfrac{c}{2}$.

17. $\dfrac{3}{a}, \dfrac{2}{b}$.

18. $\dfrac{3b}{x^2-y^2}, \dfrac{2}{x-y}$.

19. $\dfrac{3k}{x-5}, \dfrac{k}{x^2-2x-15}$.

20. $\dfrac{a}{b^2}, \dfrac{x}{a^2-b^2}, \dfrac{y}{a+b}$.

21. $\dfrac{x}{x^2-1}, \dfrac{1}{x(1-x)}$.

22. $\dfrac{a}{(a+b)^2}, \dfrac{r}{1}, \dfrac{1}{a^2-b^2}$.

23. $\dfrac{1}{b^2+5b+6}, \dfrac{1}{2(b^2+6b+9)}$.

24. $\dfrac{x-2}{x^2-2x-8}, \dfrac{x-1}{x^2-3x-10}, \dfrac{x+3}{x^2-9x+20}$.

24. Addition and Subtraction of Fractions. The following rule, which is the same in algebra as in arithmetic, will be recalled from the *First Course*, p. 151.

To add, or subtract, fractions:

1. *Reduce the fractions to equivalent fractions having their lowest common denominator (L. C. D.).*
2. *Add, or subtract, each numerator according to the sign before the fraction and write the result over the L. C. D.*
3. *Reduce the resulting fraction to its lowest terms.*

EXAMPLE. Simplify $\frac{4}{a-1}-\frac{a-2}{a+1}+\frac{3a^2}{a^2-1}$.

SOLUTION. $\frac{4}{a-1}-\frac{a-2}{a+1}+\frac{3a^2}{a^2-1}=\frac{4a+4}{a^2-1}-\frac{a^2-3a+2}{a^2-1}+\frac{3a^2}{a^2-1}$

$=\frac{4a+4-(a^2-3a+2)+3a^2}{a^2-1}=\frac{2a^2+7a+2}{a^2-1}$. *Ans.*

EXERCISES

Simplify each of the following expressions.

1. $\frac{2a}{3}+\frac{3a}{4}-\frac{a}{2}$.

2. $\frac{2x}{5}+\frac{x-3}{6}$.

3. $\frac{2r}{x}-\frac{3s}{y}$.

4. $\frac{y-4}{3}-\frac{2(1-y)}{6}+\frac{y}{8}$.

5. $\frac{3}{ab^2}-\frac{2}{ab}+\frac{4}{a^2}$.

6. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

7. $\frac{1}{2x-2}-\frac{1}{3(x+1)}+\frac{1}{3x-3}$.

8. $\frac{1}{x+y}+\frac{1}{y-x}$.

9. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

10. $\frac{r+1}{r^2-2r-8}-\frac{r}{r-4}+\frac{1}{r+2}$.

11. $\frac{1}{x^2-4x-5}-\frac{1}{x^2-6x+5}$.

12. $\frac{a+b}{a^2+2ab+b^2}-\frac{a-b}{a^2-2ab+b^2}$.

13. $\frac{x}{x-y}+\frac{y}{y-x}+1$.

14. $\frac{1}{n+4}-\frac{1}{1-n}+\frac{n-6}{n^2+3n-4}$.

15. $x-3+\frac{x^3+27}{x^2+3x+9}$.

16. $\frac{x}{x-1}\quad\frac{1+x^2}{x^2+x+1}$

25. Multiplication and Division of Fractions. Fractions are multiplied in algebra as in arithmetic by taking the product of the numerators for the numerator, and the product of the denominators for the denominator, canceling wherever possible.

EXAMPLE 1. $\dfrac{3\,a^2b}{4\,xy^2}\times\dfrac{2\,x^2y^3}{9\,ab^3}=\dfrac{3\cdot 2\,a^2bx^2y^3}{4\cdot 9\,xy^2ab^3}.$

Canceling like factors from numerator and denominator, this reduces to

$$\frac{axy}{6\,b^2}.\quad Ans.$$

EXAMPLE 2.

$$\frac{a^2+2\,a-8}{a^2+6\,a}\cdot\frac{a^2+5\,a-6}{a^2+a-12}=\frac{\cancel{(a+4)}(a-1)}{a\cancel{(a+6)}}\cdot\frac{\cancel{(a+6)}(a-1)}{\cancel{(a+4)}(a-3)}$$
$$=\frac{(a-2)(a-1)}{a(a-3)}.\quad Ans.$$

In algebra we divide one fraction by another as we do in arithmetic, by inverting the divisor and proceeding as in multiplication.

EXAMPLE.

$$\frac{a^2-b^2}{a^2+2\,ab+b^2}\div\frac{a-b}{a^2+ab}=\frac{a^2-b^2}{a^2+2\,ab+b^2}\times\frac{a^2+ab}{a-b}$$
$$=\frac{\cancel{(a-b)}\cancel{(a+b)}}{\cancel{(a+b)}\cancel{(a+b)}}\cdot\frac{a\cancel{(a+b)}}{\cancel{a-b}}=a.\quad Ans.$$

EXERCISES

Perform each of the following multiplications.

1. $\dfrac{3}{4}\cdot\dfrac{2}{5}.$

2. $\dfrac{a}{b}\cdot\dfrac{b}{c}.$

3. $\dfrac{2\,ab}{3\,xy}\cdot\dfrac{4\,x^2}{5\,a^2}.$

4. $\left(-\dfrac{6\,a}{18\,by}\right)\cdot\dfrac{10\,b^3}{8\,a^3}.$

5. $\dfrac{m^2n}{rs^2}\cdot\dfrac{mn^2}{r^3s}\cdot\dfrac{r^4s^4}{m^3n^3}.$

6. $\dfrac{a^2-b^2}{a^2+b^2}\cdot\dfrac{2}{a+b}.$

7. $\dfrac{5\,x-5\,y}{2\,x^2+4\,xy+2\,y^2}\cdot\dfrac{x-y}{5}.$

8. $\dfrac{r+2}{2\,s-3}\cdot\dfrac{s-1}{2\,r+2}\cdot\dfrac{4\,s-6}{r+2}.$

9. $\left(a+\frac{b}{c}\right)\left(a-\frac{b}{c}\right)$.

10. $\frac{a^2-b^2}{x-y}\cdot\frac{x^2-y^2}{b-a}\cdot\frac{4}{x+y}\cdot\frac{2}{a-b}$.

Perform the following divisions.

11. $\frac{4}{9}\div\frac{2}{3}$.

12. $\frac{10\,x^2}{11\,y}\div\frac{12\,x}{y^2}$.

13. $\frac{a^2+2\,a}{a^2-2\,a}\div\frac{(a+2)^2}{(a-2)^2}$.

14. $\frac{2\,ab-b^2}{a(a+b)}\div\frac{a^2-b^2}{2\,a-b}$.

15. $\frac{r^2-9\,s^2}{r^2-4\,s^2}\div\frac{r^2+rs-6\,s^2}{r^2-rs-6\,s^2}$.

Perform the indicated operations and simplify each of the following expressions.

16. $\left(\frac{x-y}{x+y}+\frac{x+y}{x-y}\right)\div\left(\frac{x-y}{x+y}-\frac{x+y}{x-y}\right)$.

17. $\left(\frac{x^2}{y^2}+\frac{2\,x}{y}+1\right)\div\left(1+\frac{x}{y}\right)$.

18. $\dfrac{\frac{3}{x}+\frac{5}{y}}{\frac{3}{x}-\frac{4}{y}}$.

19. $\left(\frac{m+2}{m}+\frac{2}{m-3}\right)\left(\frac{m}{m-2}-\frac{3}{m+3}\right)$.

20. $\left(2\,x-1+\frac{6\,x-11}{x+4}\right)\div\left(x+3-\frac{3\,x+17}{x+4}\right)$.

21. $\left(1-\frac{x-1}{x^2+6\,x+5}\right)\left(1-\frac{2}{x^2+7\,x+12}\right)$.

22. $\dfrac{\frac{1}{x+1}}{1-\frac{1}{x+1}}+\dfrac{\frac{1}{x+1}}{\frac{x}{1-x}}+\dfrac{\frac{1}{1-x}}{\frac{x}{1+x}}$.

CHAPTER V

SIMPLE EQUATIONS

26. Preliminary Considerations. Suppose we wish to divide 64 into two parts such that if one part be divided by 5 and the other by 7 the sum of the quotients shall be 10. Such a problem as this can be done only with some difficulty by arithmetic, but it is a simple task by algebra.

SOLUTION. Let x represent one part.

Then $64-x$ will be the value of the other part.

From the statement of the problem, we are to have

$$\frac{x}{5}+\frac{64-x}{7}=10.$$

Let us multiply *both* sides of this equality by 35 (as we may do without destroying it), thus clearing it of fractions. This gives

$$7x+320-5x=350.$$

Subtracting 320 from *both* sides of this last equality (as we may do without destroying it), and replacing $7x-5x$ by its value $2x$, we obtain

$$2x=30.$$

Hence (dividing both sides by 2) we have

$$x=15.$$

The two parts sought are therefore 15 and $64-15$, or 49. *Ans.*

CHECK. $\frac{15}{5}+\frac{49}{7}=3+7=10.$

A statement of equality, like any of those above, wherein a single unknown letter occurs, and occurs to no higher power than the first, is called a ***simple equation***. It is also known as a ***linear equation***, or ***an equation of the first degree***.

The process of finding the value of the unknown letter is called ***solving*** the equation.

The value of the unknown letter is called the ***solution***, or ***root***, of the equation.

27. Principles Useful in the Solving of Equations. In solving an equation we may at any point in the process add the same amount to both sides, or subtract the same amount from both, or we may multiply both by the same amount, or divide both by the same amount; as was illustrated in § 26. Derived from these are the following useful principles.

(a) *A term may be transposed (carried over) from one side (or member) of an equation to the other provided its sign is changed.*

Thus, in the equation $3x-4=2$ we may transpose the term -4 to the second member, giving $3x=2+4$, or $3x=6$. This is equivalent to adding 4 to both members of the given equation.

The solving of equations is greatly simplified by a free use of this principle of transposing terms.

Thus, in solving $3x-4=x+2$, we may transpose the -4 from the first member to the second and at the same time transpose the term x from the second member to the first, giving $-x+3x=2+4$, or $2x=6$. Therefore $x=3$. *Ans.*

(b) *A term which appears in both members of an equation may be canceled.*

Thus, by canceling the 3 from both members of the equation $2x+3=10+3$, we have simply $2x=10$, and hence $x=5$.

Note that to cancel a term in this way merely amounts to subtracting it from both members of the given equation.

(c) *The signs of all the terms in an equation may be changed.*

Thus $-5x+3=x-9$ may be written $5x-3=-x+9$.

Note that to change all signs in this manner amounts to multiplying both members of the given equation by -1.

(d) *An equation may be cleared of fractions by multiplying both members by the lowest common denominator of all the fractions.*

EXAMPLE. Solve the equation $\frac{x-2}{3}-\frac{x-3}{4}=6-\frac{x-1}{2}$.

SOLUTION. The L. C. M. of the denominators is 12.
Multiplying both members by 12,

$$4(x-2)-3(x-3)=72-6(x-1).$$

Removing parentheses (§ 7),

$$4x-8-3x+9=72-6x+6.$$

Transposing,

$$6x+4x-3x=72+6+8-9,$$

or

$$7x=77.$$

Therefore

$$x=11. \quad \textit{Ans.}$$

EXERCISES

Solve the following, using the principles stated in § 27. Check your answer for the first five.

1. $\frac{x}{3}+5=\frac{3x}{4}$.

2. $\frac{3x-1}{2}-\frac{2}{3}=\frac{x-3}{4}+\frac{6x+5}{6}$.

3. $\frac{3x}{4}-\frac{4x-2}{5}=5-\frac{5x}{8}$.

4. $\frac{1}{x}+\frac{1}{2x}=\frac{3}{8}$.

5. $\frac{x-3}{2x}-\frac{5}{12}=0$.

6. $\frac{1}{x}-\frac{x+1}{x^2}=\frac{2}{3x}$.

7. $\frac{1}{x+1}-\frac{2}{3(x+1)}=\frac{1}{12}$.

8. $\frac{1}{4-6y}-\frac{1}{2-3y}+\frac{1}{6}=0$.

9. $\frac{1}{r^2-r-2}+\frac{r+1}{r-2}=\frac{r-2}{r+1}$.

10. $\frac{x+1}{x-1}-\frac{x-3}{x+3}=\frac{8}{x}$.

11. $-\frac{3}{x-2}=\frac{1}{x+2}-\frac{2}{2-x}$.

12. $\frac{x+2}{x-4}-\frac{x-3}{x-8}=\frac{3x+8}{x^2-12x+32}$.

13. If 10 be subtracted from a certain number, three fourths of the remainder is 9. What is the number?

14. Divide 38 into two parts whose quotient is $\frac{7}{12}$.

15. Divide 96 into two parts such that $\frac{3}{4}$ of the greater shall exceed $\frac{3}{4}$ of the smaller by 6.

16. A man started on a journey with a certain sum of money. He spent $\frac{1}{4}$ of it for car fares and $\frac{1}{2}$ of it for hotel bills. When he returned home he found he had $9. How much did he start with?

17. I have $100 in one bank and $75 in another one. If I have $45 more to deposit, how shall I divide it among the two banks in order that they may have equal amounts?

18. A motor boat traveling at the rate of 12 miles an hour crossed a lake in 10 minutes less time than when traveling at the rate of 10 miles an hour. What is the width of the lake?

[HINT. *Time = Distance ÷ Rate.*]

19. A freight train goes 6 miles an hour less than a passenger train. If it goes 80 miles in the same time that a passenger train goes 112 miles, find the rate of each.

20. A tank can be filled by one pipe in 10 hours, or by another pipe in 15 hours. How long will it take to fill the tank if *both* pipes are open?

[HINT. Let x = the number of hours. Then $1/x$ = the part both can fill in 1 hour. But, $\frac{1}{10}$ = the part the first pipe can fill in 1 hour, and $\frac{1}{15}$ = the part the second pipe can fill in 1 hour. Hence we must have

$$\frac{1}{x}=\frac{1}{10}+\frac{1}{15}.\Big]$$

21. How long will it take two pipes to fill a tank if one alone can fill it in 5 hours and the other alone in 12 hours?

22. Two pipes are connected with a tank. The large one can fill it in 3 hours; the small one can empty it in 4 hours. With both pipes open, how long before the tank will fill?

23. A does a piece of work in 4 days, B in 6 days, and C in 8 days. How long will it take them working together?

24. A can do a piece of work in 16 hours, and B can do it in 20 hours. If A works for 10 hours, how many hours must B work to finish?

25. A's age is $\frac{1}{3}$ that of his father's. 12 years ago he was $\frac{1}{7}$ as old as his father. How old is each now?

26. A boat goes at the rate of 12 miles an hour in still water. If it takes as long to go 27 miles upstream as 45 miles downstream, what is the rate of the current?

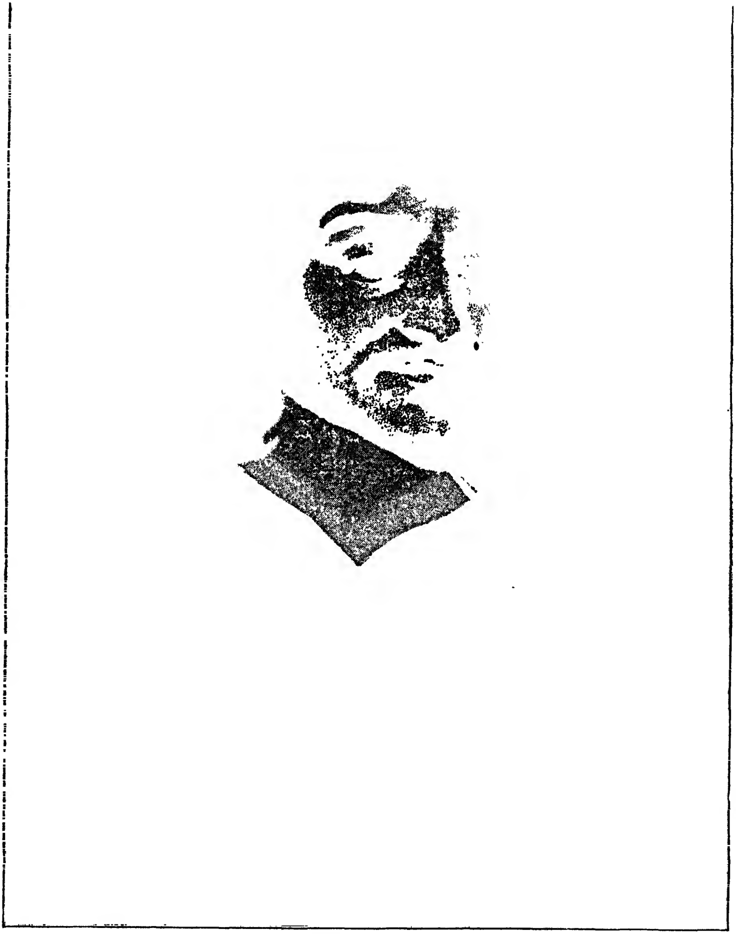
27. An aviator made a trip of 95 miles. After flying 40 miles, he increased his speed by 15 miles an hour and made the remaining distance in the same time it took him to fly the first 40 miles. What was his rate over the first 40 miles?

28. A 5-gallon mixture of alcohol and water contains 80% alcohol. How much water must be added to make it contain only 50% alcohol?

[HINT. $.50(x+5)=5\times.80$. Explain.]

29. How much water must be added to 65 pounds of a 10% salt solution to reduce it to an 8% solution?

30. A train 660 feet long running at 15 miles an hour will pass completely through the Simplon tunnel in Switzerland in $49\frac{1}{2}$ minutes. How long is the tunnel?



DESCARTES

(*René Descartes,* 1596–1650)

CHAPTER VI

GRAPHICAL STUDY OF EQUATIONS

28. Definitions. Let two lines XX' and YY' be drawn on a sheet of squared (coördinate) paper, XX' being horizontal and YY' vertical. Two such lines form a pair of ***coördinate axes.*** The point O where they intersect is called the ***origin.***

Consider any point, as P, and draw the perpendiculars PA and PB extending from P to the two axes YY' and XX' respectively. PA is then called the ***abscissa*** of P and PB is called the ***ordinate*** of P. The abscissa and ordinate taken together are called the ***coördinates*** of P.

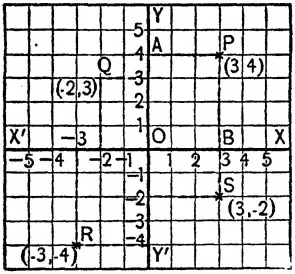


FIG. 5.

Thus the point P in the figure has its abscissa equal to 3 and its ordinate equal to 4.

All abscissas on the *right* of YY' are considered *positive,* while all abscissas on the *left* of YY' are considered *negative.*

Thus the abscissa of Q is -2; that of R is -3; that of S is $+3$.

Similarly, all ordinates *above* XX' are considered *positive,* while all ordinates *below* XX' are considered *negative.*

Thus the ordinate of Q is $+3$; that of R is -4; that of S is -2.

In reading the coördinates of a point, the abscissa is always read first and the ordinate second. Thus, in the figure, the point P is briefly referred to as *the point* (3, 4); similarly, Q is the point $(-2, 3)$; R is the point $(-3, -4)$; and S is the point $(3, -2)$, etc.

In practice, XX' is commonly called the ***x-axis***, and YY' is called the ***y-axis***.

EXERCISES

[The pupil will find it convenient to use the prepared coördinate paper such as usually may be secured at the stationery stores.]

1. Draw axes on a sheet of coördinate paper and then locate (plot) the following points:

$$(2, 4);\ (-3, -1);\ (2, -4);\ (2\tfrac{1}{2}, -3);\ (-2\tfrac{1}{4}, -2\tfrac{1}{4});$$
$$(-4, +4);\ (0, -5);\ (4, 0);\ (0, 0).$$

2. The part of the plane within the angle XOY (see figure in § 28) is called the *first quadrant*, the part within the angle YOX' is called the *second quadrant*, the part within $X'OY'$ is called the *third quadrant*, etc. Hence, state in which quadrant a point lies when

(*a*) its abscissa is positive and its ordinate negative,

(*b*) its abscissa and ordinate are both negative,

(*c*) its abscissa is negative and ordinate positive.

3. What can be said of the position of a point whose ordinate is positive; whose abscissa is negative?

4. A certain street runs due east and west. It is met by another street which runs due north and south, thus forming a "four corners." Taking the meeting place of the center-lines of the two streets as origin, and the east and north directions as positive, what are the coördinates of a flagpole which stands due northwest from the origin at a distance of 50 feet from the center-line of each road? Answer the same when the pole is 45 feet due west of the crossing point.

5. Plot the following three points and then see if it is possible to draw a straight line that will pass through all of them:

$$(1, 5)\,;\ (0, 3)\,;\ (-1, 1).$$

Do the same for the three points $(2, 3)$; $(-1, -1)$; $(5, 0)$.

29. Graph of an Equation. We have seen in Chapter V that if we have any linear equation containing a *single* unknown letter, as for example the equation $2x-1=3(x-1)$, we can always solve it; that is we can find the value of x.

Suppose now that we have a linear equation in which *two* unknown letters, x and y, appear, that is an equation in which no term contains both x and y nor any higher power of either of them than the first, as for example

$$(1) \qquad x+y=5.$$

The meaning of such an equation and the interesting facts about it are best brought out by graphical methods in ways which we shall now explain.

In the first place, it is to be observed that such an equation is satisfied by a great many *pairs of values* for x and y. For example, the pair of values $(x=1,\ y=4)$ satisfies the equation, because when we put these values for x and y respectively in the equation, it becomes $1+4=5$, which is true. Again, the same is seen to be true of the pair $(x=2,$ $y=3)$ (explain); and, similarly, the same is true of any one of the pairs $(x=\frac{1}{2},\ y=\frac{9}{2})$, $(x=6,\ y=-1)$, $(x=8,\ y=-3)$, etc. In fact, we can obtain as many such x, y pairs as we wish, each pair having the property that the x-value and the y-value *taken together* satisfy the given equation.

If we place $x=3$ in the equation above, we have $3+y=5$ and this, when solved for y, gives $y=2$. Thus $(x=3,\ y=2)$ is a pair such as mentioned above. Similarly, we can assign to x *any* value we wish (positive or negative) and find from the equation the corresponding value of y, thus forming a new pair of values of x and y.

Whenever an equation contains two (but no more) unknown letters, such as x and y, any pair of values for x and y that satisfy it is called a ***solution*** of the given equation. It follows from what has been shown above that every such equation has an indefinitely large number of solutions.

Returning to the equation $x+y=5$, let us consider again the special solutions which we noticed on page 43:

$(x=2,\ y=3),\ \ (x=\frac{1}{2},\ y=\frac{9}{2}),\ \ (x=-1,\ y=6),\ \ (x=8,\ y=-3).$

Following the ideas brought out in § 28, each of these may now be plotted as a point, using x as abscissa and y as ordinate. Upon locating these points carefully, it will be seen that they all lie on one and the same straight line, as indicated in the figure below.

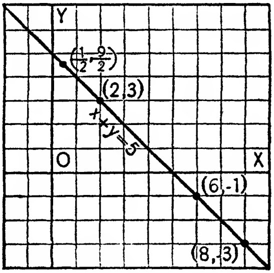


FIG. 6.

This line is called the ***graph of the equation*** $x+y=5$. It may be shown that *every* solution of the given equation gives rise when plotted to some point *upon* this line, and *vice versa*, every point upon this line has an x-value and a y-value which, when taken together, form a solution of the given equation.

30. Graph Determined from Two Points. In practice, the graph of a linear equation is drawn by locating *two* points upon it, and connecting them by a straight line.

Example. Draw the graph of the equation $5x-4y=20$.

Solution. Placing $x=0$ in the equation gives $y=-5$. Hence $(0, -5)$ is a point on the graph.

Placing $y=0$ in the equation gives $x=4$. Hence $(4, 0)$ is a point on the graph.

Plotting these two points $(0, -5)$ and $(4, 0)$ and drawing (with ruler) the straight line through them, gives the required graph.

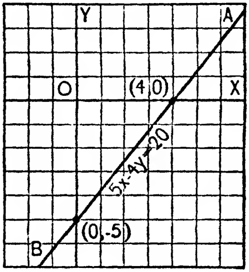


Fig. 7.

Note. As in the example just considered, it is often simplest to select as the first point the one whose abscissa is 0, and as the second point the one whose ordinate is 0. However, it is equally correct to take *any* two points whose coördinates satisfy the given equation. If the two points selected are too close to each other, it is difficult to draw the line accurately; if this happens, plot a third point on the line at a considerable distance from the first two.

EXERCISES

1. State (orally) what is true of each point on the line in the last figure.

[Hint. Its abscissa and ordinate, taken as a pair of numbers, (x, y), form a . . . .]

Draw the graph of each of the following linear equations.

2. $2x-y=4$.	4. $2x+3y=12$.	6. $4x=3y-7$.
3. $2x+y=2$.	5. $x-3y=3$.	7. $2x=3y$.

8. If a person travels at the rate of 15 miles per hour, the distance s which he will have traveled at the end of t hours is given by the formula $s = 15t$. Draw the graph of this equation, using the t-values as abscissas and the s-values as ordinates. From your figure read off (approximately) how far he will have traveled at the end of (*a*) 2 hours; (*b*) 3 hours; (*c*) $3\frac{1}{2}$ hours; (*d*) $4\frac{1}{4}$ hours.

[HINT. Take the t-axis horizontal and the s-axis vertical, and let the unit length on each be about half an inch. In order to get the diagram into relatively compact form, allow each unit on the s-axis to represent 15 miles, taking each unit on the t-axis to represent 1 hour.]

9. A boy has \$10 in the bank when he begins saving at the rate of \$3 a month, adding this amount month by month to his account. Find graphically how many months must elapse before his account will amount to \$22.

[HINT. Let A represent the amount of the account at the end of t months. Then, $A = 10 + 3t$. (Why?) Now draw the graph of this equation, using t-values as abscissas and A-values as ordinates, and taking for convenience one unit on the A-axis to represent \$2, while one unit on the t-axis represents 1 month. The problem then calls for that abscissa which goes with the ordinate $A = 22$.]

10. A boy has \$30 in the bank when be begins spending it at the rate of \$4 a month. Find graphically how long it will be before he has but \$2 left.

[HINT. Use the same letters and units as in Ex. 9.]

11. A wheel is rotating at the rate of 10 revolutions a second when the power is shut off. The wheel slows down uniformly and comes to rest at the end of 30 seconds. Make a diagram from which you can read off how many revolutions the wheel was making at any given instant after the power was shut off and use your diagram to determine how many

revolutions per second were being made at the end of (*a*) 6 seconds; (*b*) 9 seconds; (*c*) 18 seconds; (*d*) 26 seconds.

[HINT. Let r represent the number of revolutions per second at the end of t seconds. Then the conditions of the problem tell us that $r=10$ when $t=0$, and that $r=0$ when $t=30$. Thus we have two points on the graph, and we can draw the graph completely without even getting its equation.]

12. The temperature at which water freezes is 32° on the Fahrenheit scale, but it is 0° on the Centigrade scale. The temperature at which water boils is 212° on the Fahrenheit scale, but it is 100° on the Centigrade scale. Make a diagram from which you can read off the Centigrade temperature that corresponds to any given Fahrenheit temperature.

[HINT. Let F represent the Fahrenheit reading and C the Centigrade reading. Then $C=0$ when $F=32$ and $C=100$ when $F=212$. This gives two points. The graph is the straight line joining these two points.]

31. Simultaneous Equations. Suppose that, instead of having a single linear equation containing the two unknown letters x and y (as in § 29), we have *two* such equations; for example

$$x+y=6 \text{ and } 3x-2y=-2.$$

Of all the pairs of values (x, y) that will satisfy the first equation and all the pairs (x, y) that will satisfy the second equation, is there a particular pair (x, y) that will satisfy them *both* at the same time? We shall consider this question graphically.

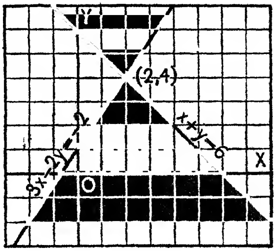


FIG. 8.

Draw the graphs of the two equations on the same sheet of coördinate paper, using the same axes throughout. The lines thus obtained are seen to intersect each other in the point (2, 4). This means that the

pair ($x=2$, $y=4$) satisfies both equations at once, since it lies on both the graphs. This pair ($x=2$, $y=4$) is therefore the pair desired, and it is the only such pair because two straight lines can intersect in but one point.

That this answer is correct is readily seen by substituting this pair of values in the given equations. Thus, with $x=2$ and $y=4$, the equations become $2+4=6$ and $6-8=-2$, which are true.

The two equations above illustrate what is known as ***a set of simultaneous equations***, and the particular pair of values ($x=2$, $y=4$) which we found would satisfy both the equations at one time, illustrates what is called the ***solution*** of the set. In general, two or more equations are said to be ***simultaneous*** if they are considered at the same time. In the present chapter we shall deal only with sets containing two unknowns, as in the preceding example.

32. Inconsistent Equations. Although two linear simultaneous equations in x and y will in general have a solution (as in § 31), there are cases where no solution can be found, and indeed none exists. For example, if we draw the graphs of the equations

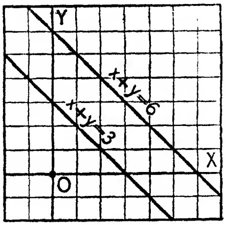


FIG. 9.

$$x+y=3, \text{ and } x+y=6,$$

we see that the lines do not intersect; in other words, they are parallel. Thus, there is no pair of values (x, y) that will satisfy both equations at once; that is, there is no solution. Such a pair of simultaneous equations is called ***inconsistent***.

EXERCISES

Determine graphically which of the following sets of simultaneous equations has a solution and which does not. In

case a solution exists, determine it and check your result by substituting it in the given equations.

1. $\begin{cases} x+2y=3, \\ 2x+y=3. \end{cases}$ 2. $\begin{cases} x+y=3, \\ 3x-y=1. \end{cases}$ 3. $\begin{cases} x+2y=5, \\ x-2y=5. \end{cases}$

4. $\begin{cases} x-2y=6, \\ 2x-4y=8. \end{cases}$ 5. $\begin{cases} 4x-y=0, \\ 3x+y=7. \end{cases}$ 6. $\begin{cases} 3x+y=-2, \\ 9x+3y=1. \end{cases}$

7. $\begin{cases} 4y-2x=5, \\ 4x+2y=5. \end{cases}$ 8. $\begin{cases} s+3t=6, \\ s-2t=6. \end{cases}$

9. A man starts at a given time and walks along a certain road at the rate of 5 miles an hour. An hour later another man starts from the same place and travels in the same direction at the rate of 10 miles an hour. Find (graphically) how far from the starting point they will meet.

[HINT. If s represents distance (in miles) traveled in t hours, the first man's motion is described by the equation $s=5t$, while the second man's motion is described by the equation $s=10(t-1)$. Now draw a pair of axes, and draw in the graphs of these two equations, using t-values as abscissas. The problem then calls for that value of s which belongs to the *intersection* of the two graphs.]

10. Use your diagram for Ex. 9 to answer the following question: After how much *time* will the two men meet?

11. A man starts and walks along a certain road at the rate of 5 miles an hour. At the same instant another man starts out at a point on the same road 15 miles distant and travels toward the first man on a bicycle at the rate of 10 miles an hour. How far from the first man's starting point will they meet? How long will it take them?

12. B and C start to save money. B has \$10 when they begin and saves at the rate of \$3 a month, while C at the start owes \$6 and saves at the rate of \$7 a month. Find graphically how soon C will be able to cancel his debt and have savings equal to B's, and how much each will then have.

CHAPTER VII

SIMULTANEOUS EQUATIONS SOLVED BY ELIMINATION

33. Elimination by Substitution. The process of combining two equations in two unknowns in such a way as to cause one of the unknowns to disappear is called ***elimination***.

We shall consider first the method called *elimination by substitution*. The process is illustrated by the following example.

EXAMPLE. Solve the system

(1) $$2x+3y=2,$$

(2) $$5x-4y=28.$$

SOLUTION. From (1),

(3) $$2x=2-3y.$$

Therefore

(4) $$x=\frac{2-3y}{2}.$$

Substituting $\frac{2-3y}{2}$ for x in (2) gives

(5) $$5\left(\frac{2-3y}{2}\right)-4y=28.$$

Clearing (5) of fractions,

(6) $$5(2-3y)-8y=56.$$

Simplifying,

(7) $$10-15y-8y=56.$$

Collecting,

(8) $$-23y=46.$$

Therefore

(9) $$y=-2.$$

Substituting -2 for y in (4) now gives

$$x=\frac{2+6}{2}=4.$$

The required solution of the system (1), (2) is, therefore,

$$(x=4,\ y=-2). \quad \textit{Ans.}$$

CHECK. Substituting $x=4$ and $y=-2$ in (1) gives

$$2(4)+3(-2),$$

which is equal to $8-6$, or 2, as (1) requires.

Substituting $x=4$ and $y=-2$ in (2) gives

$$5(4)-4(-2),$$

which is equal to $20+8$, or 28, as (2) requires.

To solve two simultaneous equations by substitution:

1. *Solve either equation for one of the unknown letters in terms of the other one.*
2. *Place the result thus obtained in the other equation and solve it.*
3. *Having thus found one of the unknown letters, substitute its value in either of the given equations and solve for the other unknown letter.*

34. Elimination by Addition or Subtraction. The only other method of elimination which we shall consider here is called *elimination by addition, or subtraction.* The process is illustrated by the following example.

EXAMPLE. Solve the system

(1) $$3x+4y=12,$$
(2) $$2x-5y=54.$$

SOLUTION. Multiplying (1) by 2,

(3) $$6x+8y=24.$$

Multiplying (2) by 3,

(4) $$6x-15y=162.$$

Subtracting (4) from (3),

(5) $$23y=-138.$$

Therefore

$$y=-6.$$

Substituting $y=-6$ in (1), $3x-24=12$, or, $3x=36$.

Therefore $x=12$.

The required solution of the system (1), (2) is, therefore,

$$(x=12,\ y=-6).\quad \textit{Ans.}$$

CHECK. Substituting 12 for x and -6 for y in (1), gives

$$3(12)+4(-6)=36-24=12, \text{ as (1) requires.}$$

Substituting 12 for x and -6 for y in (2), gives

$$2(12)-5(-6)=24+30=54, \text{ as (2) requires.}$$

NOTE. Instead of multiplying (1) by 2 and (2) by 3 and then *subtracting* them, thus eliminating x, we might just as well have multiplied (1) by 5 and (2) by 4 and *added* them, thus eliminating y. Either plan leads to the same solution for the given system.

To solve two simultaneous equations by addition or subtraction:

1. *Multiply one, or both, of the given equations by such numbers as will make the coefficients of one of the letters (say, y) numerically equal.*

2. *Subtract (or add) the two equations thus obtained, thus eliminating one of the unknown letters.*

3. *Solve the resulting equation for the letter it contains, and obtain the value of the other letter by substituting the value of the letter already found into either of the given equations.*

EXERCISES

Solve by substitution:

1. $\begin{cases} 2x+3y=12, \\ x+5y=13. \end{cases}$

2. $\begin{cases} 3x+y=7, \\ 2x-3y=-10. \end{cases}$

3. $\begin{cases} 3x-3y=-5, \\ 3x+2y=40. \end{cases}$

4. $\begin{cases} 5u+4v=0, \\ 6u-2v=-34. \end{cases}$

5. $\begin{cases} 8a+4b=5, \\ a-b=\frac{1}{4}. \end{cases}$

Solve by addition or subtraction:

6. $\begin{cases} 2x-y=5, \\ 5x+4y=19. \end{cases}$

7. $\begin{cases} 2x-5z=24, \\ 3x+5z=11. \end{cases}$

8. $\begin{cases} x+y=0, \\ 4x-3y=6. \end{cases}$

9. $\begin{cases} A+B=-9, \\ 7A-3B=7. \end{cases}$

10. $\begin{cases} 10x-3y=-6, \\ 7x+4y=8. \end{cases}$

Solve by either method:

11. $\begin{cases} \dfrac{x}{2}+\dfrac{4y}{3}=6, \\ x+\dfrac{2y}{3}=6. \end{cases}$

[HINT. First clear of fractions.]

12. $\begin{cases} \dfrac{2r+6}{5}+\dfrac{s}{3}=8, \\ \dfrac{3r-s}{12}=1. \end{cases}$

13. $\begin{cases} 2x-5y=-1, \\ \dfrac{x-y}{7}-\dfrac{3x+2y}{23}=-1. \end{cases}$

14. $\begin{cases} \dfrac{6x-8y}{3}-\dfrac{8x-20y}{11}=0, \\ x+y=\dfrac{5}{4}. \end{cases}$

15. $\begin{cases} \dfrac{3x-y}{2}+\dfrac{x+y}{3}=4, \\ \dfrac{3x+y}{11}-\dfrac{3x-y}{3}=-2. \end{cases}$

16. $\begin{cases} \dfrac{8}{x}+\dfrac{6}{y}=11, \\ \dfrac{10}{x}-\dfrac{12}{y}=4. \end{cases}$

[HINT TO EX. 16. Do not clear of fractions, but solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$.]

17. $\begin{cases} \dfrac{9}{x}+\dfrac{10}{y}=5, \\ \dfrac{15}{x}-\dfrac{30}{y}=-1. \end{cases}$

18. $\begin{cases} \dfrac{1}{x-1}+\dfrac{1}{y+1}=5, \\ \dfrac{2}{x-1}+\dfrac{3}{y+1}=12. \end{cases}$

35. Simultaneous Equations in Three Unknown Letters. We often meet with a system of three linear equations between three unknown letters. Such a system, like those already considered (§§ 33, 34), may be solved by elimination.

EXAMPLE. Solve the system

(1) $$x+y+z=6,$$
(2) $$2x-y+3z=9,$$
(3) $$x+2y-z=2.$$

SOLUTION. Eliminate one of the unknowns, say y, from (1) and (2). Thus

(4) $$3x+4z=15. \quad [(1)+(2)]$$

Eliminate the same unknown, y, from (2) and (3). Thus

$$\begin{array}{lrl} (5) & 4x-2y+6z=18. & (2)\times 2 \\ (6) & \underline{x+2y-\ z=\ 2.} & (3) \\ & 5x \qquad +5z=20, & (5)+(6) \end{array}$$

or

(7) $$x+z=4.$$

Equations (4) and (7) contain only x and z and hence may be solved for these letters, as in §§ 33, 34. Thus

$$\begin{array}{lrl} (8) & 3x+4z=15. & (4) \\ (9) & \underline{3x+3z=12.} & (7)\times 3 \\ & z=3. & (8)-(9) \end{array}$$

Substituting $z=3$ in (7), we obtain $x+3=4$. Therefore $x=1$.

Substituting $z=3$ and $x=1$ in (1), we find $1+y+3=6$. Therefore $y=2$.

The desired solution is, therefore, $(x=1,\ y=2,\ z=3)$. *Ans.*

To solve three simultaneous equations:

1. *Eliminate one of the unknown letters from any pair of the equations, then eliminate the same unknown from any other pair of the equations.*
2. *Solve the two equations thus obtained, as in* § 34.
3. *This gives two of the letters, and the third may then be found by substituting the letters already found in either of the given equations.*

EXERCISES

Solve for x, y, and z each of the following sets of equations.

1. $\begin{cases} x+2y+3z=14, \\ 2x+y+2z=10, \\ 3x+4y-3z=2. \end{cases}$

2. $\begin{cases} x-y+z=30, \\ 3y-x-z=12, \\ 7z-y+2x=141. \end{cases}$

3. $\begin{cases} x+y=9, \\ y+z=7, \\ z+x=5. \end{cases}$

4. $\begin{cases} x+y-z=0, \\ x-y=4, \\ x+z=7. \end{cases}$

5. $\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2, \\ \frac{2}{x}-\frac{1}{y}+\frac{1}{z}=7, \\ \frac{3}{x}+\frac{2}{y}+\frac{5}{z}=14. \end{cases}$

[HINT. See hint to Exercise 16, p. 53.]

APPLIED PROBLEMS

1. The sum of two numbers is 75 and their difference is 5. Find the numbers.

[HINT. Let x be one of the numbers and y the other, and form two equations.]

2. One third of the sum of two numbers is 10, while one sixth of their difference is 1. Find the numbers.

3. The perimeter of a certain rectangle is 10 inches less than 3 times the base. If the base is $4\frac{1}{2}$ times the height, what are the base and height?

4. Each base angle of a certain isosceles triangle is $66°$ more than the vertical angle. Find each angle.

5. A father's age is $1\frac{1}{2}$ that of his son. Twenty years ago his age was twice his son's. How old is each?

6. Four years ago A's age was $2\frac{1}{3}$ B's age. Four years hence A's age will be $1\frac{8}{11}$ B's age. What is the age of each?

7. A part of \$2500 is invested at 6% and the remainder at 5%. The yearly income from both is \$141. Find the amount in each investment.

*8. One sum of money is invested at 5% and another at 6%. The total yearly income from both investments is \$53.75. If the rates should be reversed, the annual income would be increased by \$2.50. Find the sums of money invested.

9. A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time could each do the work alone?

[HINT. Let x = the time in which A can do it alone, y = the time in which B can do it alone. Then the part A can do in one day is $\frac{1}{x}$, and the part B can do in one day is $\frac{1}{y}$. So the equations become $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$ and $\frac{5}{x}+\frac{26}{y}=1$. Now solve as in Ex. 16, p. 53.]

10. A and B can do a certain piece of work in 16 days. They work together for 4 days, when B is left alone and completes the work in 36 days. In what time could each do it separately?

11. A laborer agreed to stay on a farm for 100 days. For each day he worked he was to receive \$2 and board, but for each idle day he was to forfeit 75 cents for his board. When the time expired, he received \$180.75. How many days did he work?

12. An errand boy went to the bank to deposit some bills, some of them being \$1 bills and the rest \$2 bills. If there were 38 bills in all and their combined value was \$50, how many of each were there?

[HINT. Let x = the number of \$1 bills, and y = the number of \$2 bills. Then their combined value was $x+2y$ dollars.]

13. The receipts from the sale of 300 tickets for a musical recital were $125. Adults were charged 50 cents each, and children 25 cents each. How many tickets of each kind were sold?

14. A grocer wishes to make 50 pounds of coffee worth 32 cents per pound by mixing two other grades, one of which is worth 26 cents per pound and the other 35 cents per pound. How much of each must he use?

15. One cask contains 18 gallons of vinegar and 12 gallons of water; another, 4 gallons of vinegar and 12 of water. How many gallons must be taken from each so that when mixed there may be 21 gallons, half vinegar and half water?

16. Two cities are 140 miles apart. To travel the distance between them by automobile takes 3 hours less time than by bicycle, but if the bicycle has a start of 42 miles, each takes the same time. What is the rate of the automobile, and what the rate of the bicycle?

17. A boy rows 18 miles down a river and back in 12 hours. He can row 3 miles downstream while he rows but 1 mile upstream. What is his rate in still water, and what is the rate of the stream?

18. A motor boat can run r miles an hour in still water. If it went downstream for s hours and took t hours to return, what was the total distance traveled, and what was the rate of the current?

19. The sum of three numbers is 20. The sum of the first and second is 10 greater than the third, while the difference between the second and third is 6 less than the first. Find the numbers.

[HINT. Use the three letters x, y, z to represent the unknown numbers, and form three equations. Solve as in § 35.]

20. A, B, and C have certain sums of money. B would have the same as A if A gave him \$100; C would have four times as much as B if B gave him \$100; and C would have twice as much as A if A gave him \$100. How much has each?

21. I have \$90 on deposit in bank A, \$51 in bank B, and \$75 in bank C. If I have \$144 more to deposit, how shall I distribute it among the three banks so as to make the three deposits equal?

22. The perimeter of a certain rectangle is 16 feet. If the length be increased by 3 feet and the breadth by 2 feet, the area becomes increased by 25 square feet. What are the length and breadth?

23. A barrel of vinegar is to be bottled for selling and it is desired that some of the bottles be of pint size, others of quart size and others of gallon size. In order that there be 52 bottles in all, and twice as many of the pint as of the quart size, how many of each will be necessary?

[HINT. 1 barrel = 32 gallons.]

24. For any pulley block, the relation between the weight to be raised and the pull necessary to raise it is

$$pull = x + y \times weight,$$

where x and y are numbers that are different for different pulleys.

In two experiments with a certain pulley block, a weight of 100 pounds was raised by a pull of 22 pounds, and a weight of 200 pounds was raised by a pull of 42 pounds. Find the values of x and y for this pulley.

CHAPTER VIII

SQUARE ROOT

36. Definitions. The ***square root*** of a given number is the number whose square equals that number.

Thus 2 is the square root of 4 because $2^2=4$. Likewise, 3 is the square root of 9 because $3^2=9$, etc.

The square root of a number is denoted by the *radical sign* $\sqrt{}$ placed over it.

Thus $\sqrt{4}=2$, $\sqrt{9}=3$, $\sqrt{16}=4$, etc.

The process of finding the square root of a number is called *extracting* its square root.

37. Extracting Square Roots in Arithmetic. Many times we can pick out the square root of a number by inspection. Thus, $\sqrt{144}$ is seen to be 12 because $12^2 = 144$. Similarly, $\sqrt{196}=14$. But in finding the square root of a large number, such as 74,529, we cannot ordinarily determine the answer by mere inspection. The process for such a case is illustrated below, and is explained on the next page.

PROCESS.

			7′45′29	273 *Ans.*
			4	
Trial divisor	$=2\times 20$	$=40$	345	
Complete divisor	$=40+7$	$=47$	329	
Trial divisor	$=2\times 270$	$=540$	1629	
Complete divisor	$=540+3$	$=543$	1629	

EXPLANATION. First separate the number into periods of two figures each, beginning at the right. That is, in the present case, write the number in the form 7'45'29.

Find the greatest square in the left-hand period and write its root for the first figure of the required root. This gives the 2 appearing in the answer.

Square this figure (giving 4), subtract the result from the left-hand period and annex to the remainder the next period for new dividend. This gives the 345 appearing in the process.

Double the root already found, with a 0 annexed (giving 40) for a trial divisor and divide the last dividend (345) by it. The quotient (or, in some cases, the quotient diminished) forms the second figure, 7, of the required root. Add to the trial divisor the figure last found (7), giving the complete divisor (47). Multiply this complete divisor by the figure of the root last found (7), giving the 329 appearing in the process. Subtract this from the dividend, and to the remainder annex the next period for the next dividend. This gives the 1629 of the process.

Proceed as before, and continue until a new dividend equal to 0 is obtained. In the example above, this happens at once, giving 273 as the required root.

This process is the one commonly used in arithmetic, and is stated here as a review. We shall see in § 38 that a similar process may be used in extracting the square roots of expressions in algebra.

In the example just solved, the root comes out *exact* because 74,529 (whose root is being extracted) is a *perfect square* — that is, it is like one of the numbers 1, 4, 9, 16, 36, etc. If we had started with a number which was *not* a perfect square, the process would be the same except that we should not finally reach a new dividend which equals 0. In such cases, in fact, the process continues indefinitely, but if we stop it at any point, we have before us the desired root correct (decimally) up to that point. For example, in finding the square root of 550 correct to two decimal places, the process is as follows.

PROCESS.

$$
\begin{array}{r|l}
 & 5'50.00'00 \,\underline{|\, 23.45} \\
 & 4 \\
2\times 20=40 & 150 \\
40+3=43 & 129 \\ \hline
2\times 230=460 & 2100 \\
460+4=464 & 1856 \\ \hline
2\times 2340=4680 & 24400 \\
4680+5=4685 & 23425 \\ \hline
 & \quad 975
\end{array}
$$

Ans. (*correct to two decimal places*)

NOTE. In the process above we have first written 550 in the form 550.0000. If we had written it with *six* zeros, that is 550.000000, and then carried the process forward until all these were used below, we should have obtained the root correct to three decimal places instead of two. In general, the root obtained would be correct to a number of decimal places equal to half the number of zeros added.

Square roots of decimal numbers, such as 334.796, are obtained like those for whole numbers, except that in the beginning the separation of the number into periods of two figures each must be carried out *both* ways from the decimal point.

Thus 334.796 would be written 3'34.79'60. Similarly, 3.67893 would be written 3'.67'89'30. The extraction of the root is then carried out as in the process shown above.

EXERCISES

Find (by inspection or by the process shown in § 37) the square root of each of the following numbers.

1. 49.	**5.** 576.	**9.** 8281.	**13.** $\frac{4}{9}$.
2. 81.	**6.** 1444.	**10.** 15,876.	[HINT. $\frac{4}{9}=\frac{2}{3}\times\frac{2}{3}$.]
3. 64.	**7.** 4225.	**11.** 42,025.	**14.** $\frac{49}{81}$.
4. 169.	**8.** 1681.	**12.** 95,481.	**15.** $\frac{64}{225}$.

Find the square root of each of the following numbers correct to *two* decimal places.

16. 567.

17. 633.

18. 1305.

19. 17.76.

20. 13.
[HINT. Write as 13′.00′00.]

21. 2.

22. 3.

23. $\frac{3}{4}$.
[HINT. Write $\frac{3}{4}$ as .75.]

24. $\frac{11}{12}$.

38. Extracting Square Roots in Algebra.

(*a*) **Monomials.** The square root of a monomial can usually be seen by inspection.

Thus $\sqrt{36\ m^4n^2}=6\ m^2n$ (because $(6\ m^2n)^2=36\ m^4n^2$). Similarly, $\sqrt{49\ x^6y^4z^8}=7\ x^3y^2z^4$.

(*b*) **Trinomials.** If a trinomial is a perfect square, its square root can be obtained by inspection.

Thus suppose we wish to find the square root of $9\ x^2+12\ xy+4\ y^2$. This trinomial is a perfect square because its terms $9\ x^2$ and $4\ y^2$ are squares and positive, while its remaining term, $12\ xy$, is equal to $2\cdot\sqrt{9\ x^2}\cdot\sqrt{4\ y^2}$. (See § 11 (*d*), p. 20.) Hence the trinomial can be expressed in the form $(3\ x+2\ y)^2$, whence the desired square root of $9\ x^2+12\ xy+4\ y^2$ is $3\ x+2\ y$.

Similarly, $\sqrt{4s^2-4s+1}=2s-1$ because $4s^2-4s+1$ is a perfect trinomial square, and as such is factorable into

$$(2s-1)(2s-1) \text{ or } (2s-1)^2.$$

(*c*) **Polynomials.** To find the square root of a polynomial of more than three terms we may follow a process much like that employed for finding square roots in arithmetic. This is illustrated in the following example.

Example. Find the square root of $4x^4+12x^3-3x^2-18x+9$.

Process.

$$\begin{array}{lll} & 4x^4+12x^3-3x^2-18x+9 & \underline{\left|\, 2x^2+3x-3\right.} \\ & 4x^4 & \quad\quad\quad\quad \textit{Ans.} \\ \text{Trial divisor,} \quad 4x^2 & 12x^3-3x^2 & \\ \text{Complete divisor,} \quad \underline{4x^2+3x} & \underline{12x^3+9x^2} & \\ \text{Trial divisor,} \quad 4x^2+6x & -12x^2-18x+9 & \\ \text{Complete divisor,} \quad \underline{4x^2+6x-3} & \underline{-12x^2-18x+9} & \end{array}$$

Explanation. Arrange the terms of the polynomial in the descending (or ascending) powers of some letter. In the example, the arrangement is in descending powers of x.

Extract the square root of the first term, write the result as the first term of the root (giving the $2x^2$ in the answer), and subtract its square from the given polynomial (giving the $12x^3-3x^2$ in the second line of the process).

Divide the first term of the remainder by twice the root already found, used as a trial divisor. The quotient $(3x)$ is the next term of the desired root. Write this term in the root, and annex it to the trial divisor to form the complete divisor (the $4x^2+3x$ of the process).

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder (giving the $-12x^2-18x+9$ of the process).

Find the next term of the root by dividing the first term of the remainder by the first term of the new trial divisor. This gives the -3 of the answer.

Form the second complete divisor and continue in the manner above indicated until a remainder of 0 is obtained.

In the example just considered, only one letter, as x, appears. A similar process may be employed, however, in all cases by first arranging the expression in descending (or ascending) powers of some one of the letters.

For example, $4x^4+9y^6-12x^2y^3+16x^2+16-24y^3$, when arranged in descending powers of x, becomes

$$4x^4-12x^2y^3+16x^2+9y^6-24y^3+16.$$

EXERCISES

Find (by inspection or by the process shown in § 38) the square root of each of the following expressions. Check each answer by squaring it to see if the result thus obtained is the given expression.

1. $4x^4y^2$. **4.** $81a^8b^6c^{10}$. **7.** $196p^{10}q^{12}$. **10.** a^2x^{2m}.

2. $9a^6b^4$. **5.** $225m^8n^4$. **8.** $m^2n^4q^6r^{12}$. **11.** $9m^2n^{4p}$.

3. $25x^2y^4z^6$. **6.** $625r^4s^6t^2$. **9.** $529r^{10}s^{14}$. **12.** $m^{2x}n^{4y}$.

13. x^2+2x+1.

14. x^2-4x+4.

15. $4m^2+12mn+9n^2$.

16. $4x^2+4xy+y^2$.

17. $c^2-4ac+4a^2$.

18. $9m^2-6mx+x^2$.

19. $x^2+xy+\frac{1}{4}y^2$.

20. $9x^2+66x+121$.

21. $(a+b)^2-6(a+b)+9$.

22. $x^4+2x^3+3x^2+2x+1$.

23. $4x^4-12x^3+13x^2-6x+1$.

24. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.

25. $x^4-4x^3y+8x^2y^2-8xy^3+4y^4$.

[Hint. See remark at the close of § 38.]

26. $x^8+2a^6x^2-a^4x^4-2a^2x^6+a^8$.

[Hint. First arrange in descending powers of x.]

27. $9x^2-6xy+y^2+12xz-4yz+4z^2$.

28. $9x^2+25y^2+9z^2-30xy+18xz-30yz$.

29. $x^8+4x^7-3x^4-20x^5-2x^6+4+4x^2-16x+32x^3$.

39. The Double Sign of the Square Root. We know that 3 is the square root of 9 because $3^2=9$. But we also have $(-3)^2=9$. Therefore, -3 can also be regarded as a square root of 9. In other words, 9 has *two* square roots, $+3$ and -3, which are opposite in sign but otherwise the same. Similarly, 16 has the two square roots $+4$ and -4, and in general, a^2 has the two roots a and $-a$.

The double sign $\pm$ is sometimes used. Thus we say that the square root of 9 is ± 3. This is merely a brief way of saying that the two roots are $+3$ and -3.

In order to avoid all confusion, it is to be understood hereafter that the radical sign $\sqrt{\ }$ when placed over a number means the *positive* square root of that number. If it is desired to indicate the negative square root, it is done by the symbol $-\sqrt{\ }$.

Thus $\sqrt{16}$ means $+4$, while $-\sqrt{16}$ means -4. Similarly, $\sqrt{a}$ means $+\sqrt{a}$.

40. Equations Containing Radical Signs. Equations containing radical signs may often be solved by squaring each member. This is equivalent to multiplying each member by the same amount, and hence is justified by § 27.

EXAMPLE 1. Solve the equation $\sqrt{x-2}=6$.

SOLUTION. Squaring both members gives

$$x-2=36,$$

whence

$$x=38. \quad \textit{Ans.}$$

CHECK. $$\sqrt{38-2}=\sqrt{36}=6.$$

EXAMPLE 2. Solve the equation $\sqrt{x-1}-\sqrt{x-4}=1$.

SOLUTION. Transpose the $\sqrt{x-4}$ to the right; this gives

$$\sqrt{x-1}=1+\sqrt{x-4}.$$

Square both members, using Formula VI, § 10 for finding $(1+\sqrt{x-4})^2$. This gives

$$x-1=1+2\sqrt{x-4}+(\sqrt{x-4})^2,$$

or

$$x-1=1+2\sqrt{x-4}+x-4.$$

Canceling x from both sides and transposing the 1 and -4 to the left, gives

$$2=2\sqrt{x-4}, \text{ or } \sqrt{x-4}=1,$$

whence (squaring again)

$$x-4=1^2=1.$$

Therefore

$$x=5. \quad \textit{Ans.}$$

CHECK. $$\sqrt{5-1}-\sqrt{5-4}=\sqrt{4}-\sqrt{1}=2-1=1.$$

NOTE. It is especially important to check all the answers obtained for equations containing radical signs, since the process of squaring both members sometimes leads to a new equation whose roots do not all belong to the first one. Thus, if we square both members of the equation $x = 5$, we get $x^2 = 25$ and this last equation has -5 as a root as well as 5.

EXERCISES

Solve each of the following equations and check each answer.

1. $\sqrt{x+2}=4.$ 2. $\sqrt{2x+5}=3.$ 3. $\sqrt{3x-1}=2.$

4. $\sqrt{x+7}-\sqrt{x}=1.$

[HINT. First transpose one of the radicals to the right side, as in Example 2, § 40.]

5. $\sqrt{x+1}+\sqrt{x+10}=9.$ 7. $\sqrt{3x+7}-\sqrt{2x+10}=0.$

6. $\sqrt{2x+5}-\sqrt{2x+2}=1.$ 8. $\sqrt{2x+5}+\sqrt{2x-3}=4.$

9. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}.$

10. If 16 be added to 4 times a certain number, the square root of the result is 6. What is the number?

11. If 9 be added to the square of a certain number, the square root of the result is 5. What is the number?

12. The difference between the square root of a certain number and the square root of 11 less than that number is 1. Find the number.

13. Solve each of the following equations.

(a) $\sqrt{x+4}+\sqrt{x-4}=2\sqrt{x-1}.$

(b) $\sqrt{x+1}+\sqrt{x+2}-\sqrt{4x+5}=0.$

CHAPTER IX

RADICALS

41. Radicals. Suppose we have a square which we know contains exactly 2 square feet. How long is each of its four sides? In order to answer this, we naturally let x represent the desired length. Then we must have

$$x \cdot x = 2, \text{ or } x^2 = 2.$$

Therefore $\qquad x = \sqrt{2}$ ft. *Ans.*

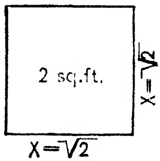


FIG. 10.

This number $\sqrt{2}$ cannot be determined exactly because it is the square root of a number, 2, which is not a perfect square. However, $\sqrt{2}$ measures a perfectly definite length, as indicated in the figure. Its value, *correct to two decimal places only*, is 1.41.

Such a number as $\sqrt{2}$ is called a ***quadratic radical***. This name is used in general to denote the indicated square root of a number.

Thus $\sqrt{3}$, $\sqrt{7}$, $\sqrt{21}$, $\sqrt{106}$, $\sqrt{213}$ are all radicals.

The word *radical* is also used in connection with other roots than square roots. Thus $\sqrt[3]{10}$ means the *cube root* of 10, that is the number whose cube is 10. Similarly, $\sqrt[4]{6}$ means the *fourth root* of 6, etc. All such numbers represent perfectly definite magnitudes, as did $\sqrt{2}$ in the figure above, even though we cannot express them *exactly* in decimal form.

In general, the nth root of any number a is written $\sqrt[n]{a}$, and this is known as a ***radical of the nth order.*** The number n is here called the ***index*** of the root, and the number a is called the ***radicand.***

NOTE. When no index is expressed, the index 2 is to be understood. Thus $\sqrt{3}$ means $\sqrt[2]{3}$.

The same definitions apply also to algebraic expressions.

Thus $\sqrt{3\,xy^2}$ and $\sqrt{a^2+b^2}$ are radicals.

42. Rational and Irrational Numbers. Surds. The positive and negative integers and fractions, and zero, are called ***rational*** numbers. If an indicated root of a number cannot be extracted exactly, that is, cannot be expressed exactly as one of these *rational* numbers, it is called a ***surd.*** Any positive or negative number that is not rational is called ***irrational.***

Thus $\sqrt{3}$, $\sqrt[3]{10}$, $\sqrt{6}$ are surds; but $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt{\frac{9}{16}}$ are rational, since $\sqrt{9}=3$, $\sqrt[3]{8}=2$, $\sqrt{\frac{9}{16}}=\frac{3}{4}$.

EXERCISES

Determine which of the following radicals are surds; and state the index and the radicand of each.

1. $\sqrt{7}$. **2.** $\sqrt{8}$. **3.** $\sqrt{16}$. **4.** $\sqrt{\frac{6}{13}}$. **5.** $\sqrt{\frac{25}{36}}$. **6.** $\sqrt{.5}$.

7. $\sqrt[3]{-8}$.

[HINT. $-8=(-2)^3$.]

8. $\sqrt[3]{15}$. **10.** $\sqrt[4]{5}$. **12.** $\sqrt[5]{20}$. **14.** $\sqrt{3\,x^2y^2}$.

9. $\sqrt[3]{27}$. **11.** $\sqrt[4]{16}$. **13.** $\sqrt[5]{32}$. **15.** $\sqrt[3]{8\,m^6(a+b)^9}$.

43. Value of Radicals. Use of Table. To determine the value of a radical correct to two or more decimal places usually calls for a rather long process. (See § 37, p. 61.) In order to save time and labor, the values of those radicals which are needed most in ordinary life (the square and cube

roots) have been printed in a table and placed for convenient reference at the end of this book. For the sake of completeness, the second and third *powers* of numbers are also printed in the table. Just how to use this table is described on page 275, which the pupil should now read carefully. Below are a few illustrative examples.

EXAMPLE 1. Find $\sqrt{7}$, using the table.

SOLUTION. The top number in the *third* column on page 290 (table) gives $\sqrt{7}=2.64575$. This value is correct up to the last decimal figure given, that is to the fifth place. Thus the answer may be written $\sqrt{7}=2.64575^+$, the sign $+$ indicating that this value for $\sqrt{7}$ is correct up to the last decimal place stated.

EXAMPLE 2. Find $\sqrt[3]{7}$, using the table.

SOLUTION. The top number in the *sixth* column on page 290 (table) gives $\sqrt[3]{7}=1.91293^+$. *Ans.*

EXAMPLE 3. Find $\sqrt{70}$.

SOLUTION. The top number in the *fourth* column on page 290 (table) gives $\sqrt{70}=8.36660^+$. *Ans.*

EXAMPLE 4. Find $\sqrt[3]{70}$.

SOLUTION. $\sqrt[3]{70}=4.12129^+$, from the seventh column, page 290.

EXAMPLE 5. Find $\sqrt[3]{700}$.

SOLUTION. $\sqrt[3]{700}=8.87904^+$, from the eighth column, page 290.

EXERCISES

By means of the table, determine the approximate values of the following radicals.

1. $\sqrt{6}$. 2. $\sqrt{60}$. 3. $\sqrt[3]{6}$. 4. $\sqrt[3]{60}$. 5. $\sqrt[3]{500}$.

6. $\sqrt{61}$.

[HINT. $61=6.10\times 10$. So use the information given for 6.10 in the table.]

7. $\sqrt[3]{61}$. 8. $\sqrt{92}$. 9. $\sqrt[3]{920}$. 10. $\sqrt{7.8}$ 11. $\sqrt{7.82}$

12. $\sqrt{78.2}$ 13. $\sqrt{782}$.

SOLUTION. $782 = 7.82 \times 100$. Therefore $\sqrt{782}$ is the same as $\sqrt{7.82}$ except that the decimal point in the root must be moved *one* place farther to the right. (See p. 275.) Now, $\sqrt{7.82} = 2.79643$ (table), so $\sqrt{782} = 27.9643$. *Ans.*

14. $\sqrt{561}$.

[HINT. See Solution of Ex. 13.]

15. $\sqrt{779}$. 16. $\sqrt{895}$.

17. $\sqrt[3]{6120}$.

[HINT. $6120 = 6.12 \times 1000$. (See p. 276.)]

18. $\sqrt[3]{5340}$.

19. $\sqrt{.67}$

[HINT. $.67 = \frac{1}{10} \times 6.7$ (Now see p. 276.)]

20. $\sqrt{.0676}$

APPLIED PROBLEMS

Use the tables in working the following problems.

1. If the sides of a right triangle are 3 inches and 2 inches long, respectively, what is the length of the hypotenuse?

[HINT. If x be the hypotenuse, then $x^2 = 3^2 + 2^2 = 13$.]

2. A baseball diamond is a square 90 feet on a side. How far is it from home plate to second base?

3. If the diagonal of a square is 13 feet long, how long is each side?

4. The dimensions of a certain rectangular field are 103 feet by 337 feet. In going from one corner to the opposite corner, how much shorter is it to go by the diagonal than to go around?

5. How long must the radius of a circle be in order that the area be 12 square inches? (See Ex. 14 (*b*), p. 6. Take $\pi = 3\frac{1}{7}$.)

6. What is the length of the edge of a cube if the volume is 357 cubic inches?

7. If the volume of a sphere is 440 cubic inches, how long is its radius? (See Ex. 14 (*e*), p. 6.)

8. In the accompanying figure how long should the radius of the inner semicircle be in order that the area inclosed may be 132 square feet?

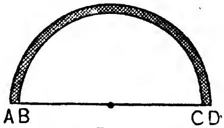


FIG. 11.

9. The area A of a triangle in terms of its three sides, a, b, and c, is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$. The sides of a triangle are respectively 6 inches, 7 inches, and 9 inches long. What is the area?

10. Two circular cones have altitudes, h, which are the same, but their bases have different radii. What is the ratio of the longer radius to the shorter if the volume of the one cone is three times that of the other?

[HINT. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where h represents the altitude and r is the radius of the base.]

44. Simplification of Radicals. We know that the square root of the product of two numbers is the same as the product of their square roots. For example, $\sqrt{4 \times 25}$ is the same as $\sqrt{4} \times \sqrt{25}$, because both are equal to 10 (the first being $\sqrt{100}$, or 10, and the second being 2×5, or 10). In the same way, $\sqrt[3]{8 \times 3} = \sqrt[3]{8} \times \sqrt[3]{3}$, or simply $2\sqrt[3]{3}$. In fact, we have the following general formula.

Formula I. $$\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}.$$

Again, $\sqrt{\frac{4}{9}}$ is the same as $\dfrac{\sqrt{4}}{\sqrt{9}}$ because both are equal to $\frac{2}{3}$. (Explain.) Similarly, $\sqrt[3]{\frac{5}{8}}$ may be written $\dfrac{\sqrt[3]{5}}{\sqrt[3]{8}}$, or $\dfrac{\sqrt[3]{5}}{2}$. So in general we have the following formula.

Formula II. $$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Formulas I and II enable us to simplify many radical expressions, as is illustrated in the following examples.

Example 1. Simplify $\sqrt{63}$.

Solution. Using Formula I, we have

$$\sqrt{63}=\sqrt{9\times 7}=\sqrt{9}\times\sqrt{7}=3\sqrt{7}. \quad \textit{Ans.}$$

Example 2. Simplify $\sqrt[3]{32}$.

Solution. $\sqrt[3]{32}=\sqrt[3]{8\times 4}=\sqrt[3]{8}\times\sqrt[3]{4}=2\sqrt[3]{4}$. *Ans.*

Example 3. Simplify $\sqrt{\frac{8}{27}}$.

Solution. $$\sqrt{\frac{8}{27}}=\frac{\sqrt{8}}{\sqrt{27}}=\frac{\sqrt{4\times 2}}{\sqrt{9\times 3}}=\frac{\sqrt{4}\times\sqrt{2}}{\sqrt{9}\times\sqrt{3}}=\frac{2\sqrt{2}}{3\sqrt{3}}. \quad \textit{Ans.}$$

Example 4. Simplify $\sqrt{20\,a^6}$.

Solution. $\sqrt{20\,a^6}=\sqrt{4\,a^6\times 5}=\sqrt{4\,a^6}\times\sqrt{5}=2\,a^3\sqrt{5}$. *Ans.*

Example 5. Simplify $\sqrt[3]{\dfrac{72\,x^2y^6}{z^6}}$.

Solution.

$$\sqrt[3]{\frac{72\,x^2y^6}{z^6}}=\frac{\sqrt[3]{(8\,y^6)\times(9\,x^2)}}{\sqrt[3]{z^6}}=\frac{\sqrt[3]{8\,y^6}\times\sqrt[3]{9\,x^2}}{z^2}=\frac{2\,y^2\sqrt[3]{9\,x^2}}{z^2}. \quad \textit{Ans.}$$

Note. It will be observed that in each of the examples above the process of simplification consists in removing from under the

radical sign the largest factor of the radicand which is a perfect square, or perfect cube, as the case may be. Thus, in Example 1, *the radicand, 63, was first broken up into factors in such a way that* 9 (which is a perfect square) appears clearly to the eye. Similarly, in Example 2 (where we are dealing with a *cube* root) we first write the radicand, 32, in a form which brings out conspicuously its factor 8, which is a perfect cube. The first step in all such examples is, therefore, to get the radicand properly broken up into factors. This requires good judgment, but becomes very easy after slight practice and experience.

EXERCISES

1. By Formula I, p. 72, we have $\sqrt{20}=\sqrt{4\times 5}=\sqrt{4}\times\sqrt{5}=2\sqrt{5}$. Look up the values of $\sqrt{20}$ and $\sqrt{5}$ in the table and thus prove that $\sqrt{20}$ *is* the same as $2\sqrt{5}$.

2. Show that Formula I, p. 72, gives $\sqrt{54}=3\sqrt{6}$ and verify the correctness of this result by use of the table, as in Ex. 1.

Simplify each of the following radicals. (See Note in § 44.)

3. $\sqrt{18}$.

4. $\sqrt{24}$.

5. $\sqrt{112}$.

6. $\sqrt{125}$.

7. $\sqrt{108}$.

8. $\sqrt[3]{32}$.

9. $\sqrt[3]{54}$.

10. $\sqrt[3]{81}$.

11. $\sqrt{\frac{72}{75}}$.

[Hint. First use Formula II, § 44.]

12. $\sqrt[3]{\frac{24}{135}}$.

13. $\sqrt[4]{96}$.

14. $\sqrt[4]{162}$.

15. $\sqrt{36\,a^5b^3}$. $6\,a^2b\sqrt{ab}$. *Ans.*

16. $\sqrt{81\,m^5n^7}$.

17. $\sqrt{27\,x^3y^3z^3}$.

18. $\sqrt{4(a+b)^3}$.

19. $\sqrt[3]{27\,x^4y^3z^2}$.

20. $\sqrt{\dfrac{16\,h^2k^4}{s^3t}}$.

21. $\sqrt[3]{\dfrac{16\,h^2k^4}{s^3t}}$.

22. $\sqrt{\dfrac{3(a+b)^2c^2d}{4(a^2-b^2)}}$

Write each of the following in a form having no coefficient outside the radical sign.

23. $2\sqrt{3}$.

SOLUTION. By Formula I, § 44, $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{12}$. *Ans.*

24. $2\sqrt{2}$. **25.** $5\sqrt{5}$. **26.** $2\sqrt[3]{2}$.

27. $\frac{1}{3}\sqrt{2}$.

[HINT. Write $\frac{1}{3}\sqrt{2}=\frac{\sqrt{2}}{3}=\frac{\sqrt{2}}{\sqrt{9}}$, and apply Formula II, § 44.]

28. $\frac{1}{3}\sqrt{10}$. **33.** $2\,g\sqrt{x-y}$. **38.** $\frac{3}{4}\sqrt{3}$.

29. $a\sqrt{b}$. $\sqrt{a^2b}$. *Ans.* **34.** $m\sqrt[3]{n}$. **39.** $\frac{2}{3}\sqrt[3]{3}$.

30. $x\sqrt{3\,y}$. **35.** $2\,a\sqrt{a^2-b^2}$. **40.** $\frac{2}{3}\sqrt{\frac{9}{4}}$.

31. $mn\sqrt{5\,mn}$. **36.** $\frac{a\sqrt{x^3}}{x\sqrt{a^3}}$. **41.** $(a-b)\sqrt{2\,c}$.

32. $5\,mn\sqrt{mn}$. **37.** $\frac{a}{b}\sqrt{\frac{b}{a}}$. **42.** $(a-b)\sqrt[3]{2\,c}$.

45. Addition and Subtraction of Similar Radicals. Whenever two radicals having the same index have also the same radicand (or can be given the same radicand by simplification) they are called ***similar radicals***.

Thus $2\sqrt{2}$ and $3\sqrt{2}$ are similar radicals; so also are $\sqrt{2}$ and $\sqrt{32}$ since the last of these may be simplified into $4\sqrt{2}$, by § 44. Likewise, $\sqrt{3\,a^2x}$ and $\sqrt{3\,b^2x}$ are similar, being equal, respectively, to $a\sqrt{3\,x}$ and $b\sqrt{3\,x}$, thus coming to have the *same* radicand.

Whenever similar radicals are added or subtracted, the result may be expressed in a single term.

Thus $4\sqrt{3}+5\sqrt{3}=(4+5)\sqrt{3}=9\sqrt{3}$. *Ans.*

Again, $3\sqrt{32}-2\sqrt{8}=3\times4\sqrt{2}-2\times2\sqrt{2}=12\sqrt{2}-4\sqrt{2}$
$=(12-4)\sqrt{2}=8\sqrt{2}$. *Ans.*

Likewise,

$$2\sqrt{4\,a^2b}+\sqrt{9\,a^2b}-\sqrt{16\,a^2b}=2\cdot2\,a\sqrt{b}+3\,a\sqrt{b}-4\,a\sqrt{b}$$
$$=(4\,a+3\,a-4\,a)\sqrt{b}=3\,a\sqrt{b}.\quad \textit{Ans.}$$

NOTE. The pupil is especially warned that in general we *cannot* write $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$. Thus when $a=4$, $b=9$ this would give $\sqrt{13}=2+3=5$, which is clearly false. Similarly, we *cannot* write $\sqrt{a-b}=\sqrt{a}-\sqrt{b}$. Thus $\sqrt{25-9}=\sqrt{16}=4$, but $\sqrt{25}-\sqrt{9}=5-3=2$.

EXERCISES

Combine the following radicals.

1. $\sqrt{8}+\sqrt{18}+\sqrt{32}$. Check your answer by use of the table; that is show that $\sqrt{8}+\sqrt{18}+\sqrt{32}$, as computed by the table, has the same value as your answer, similarly computed.

2. $\sqrt{108}+\sqrt{27}-\sqrt{75}$. Check your answer as in Ex. 1.

3. $\sqrt[3]{128}+\sqrt[3]{16}-\sqrt[3]{54}$. Check as in Exs. 1 and 2.

4. $\sqrt{72}+\sqrt{32}-\sqrt{50}$.

5. $\sqrt{\frac{3}{4}}+\sqrt{\frac{27}{16}}+\sqrt{\frac{75}{9}}$.

6. $\sqrt[3]{24}+\sqrt[3]{81}+\sqrt[3]{192}$.

7. $\sqrt{32\,a^2}-\sqrt{8\,a^2}+\sqrt{18\,a^2}$.

8. $\sqrt[3]{16\,a^3b^3}+\sqrt[3]{54\,a^3b^3}$.

9. $\sqrt{32\,a}-\sqrt{8\,a}+\sqrt{18\,a}$.

10. $\sqrt{2(x-y)^2}+\sqrt{8(x-y)^2}+\sqrt{18(x-y)^2}$.

11. $\sqrt{2(x-y)}+\sqrt{8(x-y)}+\sqrt{18(x-y)}$.

12. $\sqrt{\frac{1}{2}}+\sqrt{12\frac{1}{2}}+\sqrt{\frac{1}{8}}+\sqrt{1\frac{1}{8}}$.

13. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+3\sqrt{27}$.

14. $\sqrt[3]{\frac{8}{5}}+\sqrt[3]{\frac{1}{5}}+\sqrt[3]{5\frac{2}{5}}$.

15. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}$.

46. Multiplication of Radicals. We may multiply a radical, as $\sqrt[n]{a}$, by another of the same index, as $\sqrt[n]{b}$, by formula I of § 44. If $n=2$, this gives as an important case

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}.$$

EXAMPLE 1. Find the product of $\sqrt{2}$ and $\sqrt{18}$.

SOLUTION. $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$. *Ans.*

EXAMPLE 2. Multiply $\sqrt{3}+\sqrt{5}$ by $2\sqrt{3}-\sqrt{5}$.

SOLUTION.

$$\begin{array}{l}
\sqrt{3}+\ \sqrt{5} \\
\underline{2\sqrt{3}-\ \sqrt{5}\quad} \\
2 \cdot 3+2\sqrt{15} \\
\qquad -\ \sqrt{15}-5 \\
\quad 6+\ \sqrt{15}-5=1+\sqrt{15}. \quad \textit{Ans.}
\end{array}$$

EXAMPLE 3. Multiply $\sqrt{a}+\sqrt{a-b}$ by $\sqrt{a}-\sqrt{a-b}$.

SOLUTION.

$$\begin{array}{l}
\sqrt{a}+\sqrt{a-b} \\
\underline{\sqrt{a}-\sqrt{a-b}\qquad} \\
a+\sqrt{a^2-ab} \\
\underline{\quad -\sqrt{a^2-ab}-(a-b)} \\
a \qquad\qquad\quad -(a-b)=a-a+b=b. \quad \textit{Ans.}
\end{array}$$

EXERCISES

Find the following products, simplifying results as far as possible.

1. $\sqrt{3} \cdot \sqrt{27}$.
2. $\sqrt{8} \cdot \sqrt{12}$.
3. $\sqrt{6} \cdot \sqrt{4}$.
4. $\sqrt{7} \cdot \sqrt{9}$.
5. $\sqrt{10} \cdot \sqrt{3} \cdot \sqrt{2}$.

$\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{2}}$.

7. $\sqrt{7} \cdot \sqrt{\frac{1}{7}}$.
8. $\sqrt{.1} \cdot \sqrt{.01}$.
9. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$.
10. $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$
11. $(\sqrt{6}-\sqrt{3})^2$.
12. $(\sqrt{7}-1)^2$.

$(3\sqrt{3}+2\sqrt{5})(\sqrt{3}-3\sqrt{5})$.

$(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3})$.

$\sqrt{a} \cdot \sqrt{a^3}$.

$\sqrt{ab} \cdot \sqrt{a^3b^3}$.

$\sqrt{x^2y} \cdot \sqrt{xy}$.

18. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.
19. $(\sqrt{a}-\sqrt{b})^2$.
20. $(\sqrt{2x}+\sqrt{9y})^2$.

21. $(2\sqrt{x}+3\sqrt{y})(3\sqrt{x}-2\sqrt{y})$.

22. Find the value of x^2-4x-1 if $x=2+\sqrt{5}$.

23. Find the value of x^2+3x-2 if $x=\dfrac{\sqrt{17}-3}{2}$.

24. Does $\sqrt{3}+2$ satisfy the equation $x^2-4x+1=0$; that is, is the equation true when $x=\sqrt{3}+2$? Answer the same question when $x=\sqrt{3}+\sqrt{2}$.

47. Division of Radicals. We may divide one radical, as $\sqrt[n]{a}$, by another of the same index, as $\sqrt[n]{b}$, by the Formula II of § 44. If $n=2$, this gives as an important special case

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}.$$

EXERCISES

Express each of the following quotients as a fraction under one radical sign, and reduce your answer to simplest form.

1. $\dfrac{\sqrt{15}}{\sqrt{3}}$.

SOLUTION. $\dfrac{\sqrt{15}}{\sqrt{3}}=\sqrt{\dfrac{15}{3}}=\sqrt{5}$. *Ans.*

2. $\dfrac{\sqrt{30}}{\sqrt{5}}$.

3. $\dfrac{\sqrt{3}}{\sqrt{12}}$.

4. $\dfrac{\sqrt{2}}{\sqrt{6}}$. 5. $\dfrac{\sqrt{6}}{\sqrt{3}}$. 6. $\dfrac{\sqrt{x^5}}{\sqrt{x^3}}$. 7. $\dfrac{\sqrt{a^2-b^2}}{\sqrt{a+b}}$. 8. $\dfrac{\sqrt{3(x^2-1)}}{\sqrt{12(x-1)}}$.

9. $\sqrt{x^2+x-20}\div\sqrt{x+5}$.

10. $\sqrt{81+9a^2+a^4}\div\sqrt{a^2-3a+9}$.

11. $\sqrt{x^6-y^6}\div\sqrt{x^3+y^3}$.

12. $\sqrt{x^7+2x^6-2x^4+2x^2-1}\div\sqrt{x+1}$.

13. $\sqrt{r^{2n}+11r^n+30}\div\sqrt{r^n+6}$.

CHAPTER X

QUADRATIC EQUATIONS

48. Quadratic Equation. An equation which contains the unknown letter to the second (but no higher) power is called a ***quadratic equation***, or briefly, a ***quadratic***.

Thus the equations $2x^2-4x=1$ and $\frac{1}{2}x^2+x=-3$ are quadratics, but $2x-3=0$ and $4x^3-5x^2+x=2$ are not.

49. Pure Quadratic. When the quadratic contains the second power only of the unknown letter, it is called a ***pure quadratic***.

Thus $2x^2-27=0$ and $ax^2=bc$ are pure quadratics, but $x^2-4x=2$ and $x^2+bx+c=0$ are not.

50. Affected Quadratic. When the quadratic contains both the first and second powers of the unknown letter, it is called an ***affected quadratic***.

Thus $x^2+3x=7$ and $x^2+2ax=a^2$ are affected quadratics, but $2x^2-7=0$ and $5x^2-16a^2b^2=c^2$ are not.

51. Solution of Pure Quadratics. The following example ·ll suffice to show how the solution of any pure quadratic .y be obtained.

'XAMPLE. Solve $2x^2-30=0$.

›LUTION. Transposing and dividing through by 2 gives $x^2=15$. king the square root of both members gives $x=\pm\sqrt{15}$. *Ans.* get the approximate value of $\sqrt{15}$, we may consult the table, we find $\sqrt{15}=3.87298^+$.

answer may, therefore, be written in the form $x=\pm 3.87298^+$.

'K. $2(\sqrt{15})^2-30=2\times 15-30=30-30=0$, as required.

$2(-\sqrt{15})^2-30=2\times 15-30=30-30=0$, as required.

NOTE. Strictly speaking, when we extract the square root of both members of the equation $x^2=15$ we get $\pm x = \pm\sqrt{15}$. But to say that $-x = \pm\sqrt{15}$ means the same as $+x = \pm\sqrt{15}$, so it suffices to write simply $x = \pm\sqrt{15}$ to cover all cases.

An examination of the example above shows that we have the following rule.

To solve a pure quadratic, solve for x^2, then take the square root of the result. There will be **two** *solutions, the one being the negative of the other.*

EXERCISES

Solve each of the following equations, checking your answer for the first five. If you meet with a radical, find its approximate value by use of the table.

1. $x^2-81=0.$
2. $3x^2-192=0.$
3. $4x^2+8=10x^2-16.$
4. $3x^2-15=0.$
5. $3x^2-16=0.$
6. $3x^2-17=0.$
7. $\dfrac{x^2}{7}-\dfrac{x^2}{8}=19.$
8. $\dfrac{x-8}{6}=\dfrac{6}{x+8}.$
9. $\dfrac{x+3}{x-3}+\dfrac{x-3}{x+3}=4.$
10. $(x+1)^2-2(x+1)=4.$
11. $\dfrac{3}{x+5}=1+\dfrac{1}{2x-5}.$
12. $\dfrac{1}{x^2-1}-\dfrac{1}{1-x}=\dfrac{7}{8}+\dfrac{1}{x+1}.$
13. $\dfrac{x+3}{2x+1}+\dfrac{2x-1}{x-3}=0.$
14. $\dfrac{2x^2+1}{x+1}-\dfrac{2x^2-1}{x}=2.$
15. $\dfrac{x^2-x+2}{x-2}-\dfrac{x^2+x-3}{x+3}=$
16. $\sqrt{2x^2+x}=x+\frac{1}{2}.$
    [HINT. See § 40.]
17. $\sqrt{(x+3)(x-5)}=\sqrt{}$
18. $\sqrt{25-6x}+\sqrt{25-}$

APPLIED PROBLEMS

1. What numbers are equal to their own reciprocals?

2. One side of a right triangle measures 3 inches and the hypotenuse measures 7 inches. Find (approximately) the length of the other side.

[HINT. Work by algebra, making use of the principle that in any right triangle the square of the hypotenuse equals the sum of the squares of the two sides.]

3. What is the length of the longest umbrella than can be placed in the bottom of a trunk the inside of which is 33 inches long by 21 inches wide?

4. A certain square has a side which is three times as long as the side of another square. If the difference of their areas is 72 square feet, how long is the side of each?

5. Find the mean proportional between 25 and 9; also that between 17 and 21. In what particular is the latter one essentially different from the first one?

6. It is proved in geometry that whenever a perpendicular is drawn from a point on a semicircle to the base, PQ in Fig. 12, its length is a mean portional between the segments AP PC of the base; that is,

$$\frac{AP}{PQ}=\frac{PQ}{PC}.$$

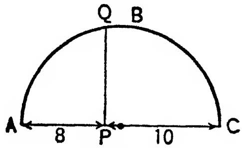


FIG. 12.

$P=8$ inches and $PC=10$ inches, ng is PQ?

termine the formula for

he side of the square whose area is a.

e radius of the circle whose area is a.

(*c*) The radius of the sphere whose area is a.

[HINT. See Ex. 14, p. 6.]

(*d*) The diameter of the base of a circular cone whose volume is v and whose altitude is h.

[HINT. The volume of a circular cone is equal to the area of its base multiplied by $\frac{1}{3}$ its altitude, *i.e.*, $V=\frac{1}{3}\pi r^2 h$.]

8. The distance s, measured in feet, through which an object falls in t seconds when dropped vertically downward is given by the formula $s=\frac{1}{2}gt^2$, where $g=32$ (approximately). Hence, determine (approximately) how long it will take a stone to drop to the bottom of a mine 300 feet deep.

9. One of the sides of a certain triangle is m units long. What is the formula for the corresponding side of a similar triangle whose area is n times as great?

[HINT. It is shown in geometry that if two geometric figures are similar; that is, have the same shape but are of different sizes, then the square of any line in the first figure is to the square of the corresponding line in the second figure as the area of the first figure is to the area of the second.]

10. A map of the United States is uniformly enlarged in such a way as to cover twice as much area on the paper a before. By what factor should the scale of the map be no multiplied?

11. Find three consecutive integers such that the sq of the second plus the product of the other two equals 3

[HINT. Let x be the second integer. Then the first and integers will be $x-1$ and $x+1$ respectively.]

52. Solution of Affected Quadratics by Factoring. familiar principle of arithmetic that the product numbers is zero if either of the numbers is zero; th either factor is zero.

For example $2\times 0=0$, $0\times 4=0$, $(-3)\times 0=0$, etc.

This principle is frequently used to solve affected quadratic equations.

EXAMPLE. Solve by factoring the equation $x^2+8x=48$.

SOLUTION. Transposing all terms to the left, we have

$$x^2+8x-48=0.$$

Factoring,

$$x^2+8x-48=(x-4)(x+12). \qquad [\S\ 11\ (c)]$$

Thus the given equation becomes

$$(x-4)(x+12)=0.$$

This equation will be satisfied, according to the principle mentioned above, whenever the factor $x-4$ equals zero or the factor $x+12$ equals zero, that is in case $x-4=0$ or $x+12=0$. Solving these two simple equations gives $x=4$ and $x=-12$, which must therefore be the desired solutions.

CHECK. When $x=4$ the left side of the original equation becomes $4^2+8\times4$, which reduces to $16+32=48$, as the equation demands. When $x=-12$ we have in like manner

$$(-12)^2+8\times(-12)=144-96=48.$$

We thus have the following rule.

To solve quadratics by factoring:

1. *Transpose all terms to the left so as to have 0 on the right.*
. *Factor the left member of the resulting equation.*
Place each factor equal to 0 and solve the resulting simple ...*tions. The two results are the solutions required.*

EXERCISES

...e each of the following equations by factoring, checking ...swer in the first five.

$x^2-7x+10=0$. 3. $x^2+8x=-15$.

$x^2-5x=-6$. 4. $x^2+7x-30=0$.

5. $x^2+4x=0$.

6. $1-6x^2=x$.

7. $5x+24=x^2$.

8. $x^2-1=3(x+1)$.

9. $x-\dfrac{5}{x+4}=0$.

10. $\dfrac{x^2}{x-2}=-\dfrac{4}{x-2}-5$.

11. $\dfrac{x-2}{x+2}-\dfrac{x+3}{x-3}=-\dfrac{20}{3}$.

12. $(x+4)(x-2)=11(x-2)$.
[Hint. Write as
$$(x-2)[(x+4)-11]=0,$$
then apply the principle in § 52.]

13. $3(x+1)(x-3)+4(x-3)=0$.

14. $(x+1)^2+3(x+1)+2=0$.
[Hint. Solve first for $x+1$.]

53. Solution of Any Quadratic by Completing the Square. We often meet with a quadratic, such as $x^2+7x-5=0$, which we cannot solve as in § 52 by factoring. The difficulty here is that we cannot factor readily x^2+7x-5. However, this quadratic and *all* others (whether solvable by factoring or not) can be solved by a certain process known as *completing the square.* How this is done will be best understood from a careful study of the following examples.

Example 1. Solve $x^2+6x=16$.

Solution. The first member of this equation, or x^2+6x, would become a trinomial square [§ 11(*d*)] if 9 were added to it. Our first step, therefore, is to add 9 to *both* members of the given equa tion, thus "completing the square" in the first member and givir us the equation
$$x^2+6x+9=25,$$
or
$$(x+3)^2=25.$$

Taking the square root of both members of the last equati now an easy process and gives
$$x+3=\pm 5.$$

Therefore we must either have $x+3=5$, or $x+3=-5$. Solving the last two equations gives as the desired sc $x=2$ and $x=-8$. *Ans.*

Check. Substituting 2 for x in the first member of th equation gives $2^2+6\times 2$, which reduces to $4+12=16$, as Similarly, with x equal to -8, the first member of the gi tion becomes $(-8)^2+6\times(-8)$, or $64-48$, which reduc required.

EXAMPLE 2. Solve $x^2-8\,x+14=0$.

SOLUTION. Transposing, $x^2-8\,x=-14$.

Completing the square by adding 16 to both sides gives

$$x^2-8\,x+16=2, \text{ or } (x-4)^2=2.$$

Taking the square root of both members,

$$x-4=\pm\sqrt{2}.$$

Solving the last two equations,

$$x=4+\sqrt{2} \text{ and } x=4-\sqrt{2}. \quad \textit{Ans.}$$

CHECK. With $x=4+\sqrt{2}$, the first member of the given equation becomes $(4+\sqrt{2})^2-8(4+\sqrt{2})+14$. By Formula VI of § 10 this may be written

$$(16+8\sqrt{2}+2)-8(4+\sqrt{2})+14, \text{ or } 16+8\sqrt{2}+2-32-8\sqrt{2}+14.$$

Here the $8\sqrt{2}$ and the $-8\sqrt{2}$ cancel, while the rest of the expression (namely $16+2-32+14$) reduces to 0, as required.

Likewise, when x has its other value, namely $x=4-\sqrt{2}$, the first member may be shown to become 0.

NOTE. Since the solutions obtained above for Example 2 contain the surd $\sqrt{2}$, they cannot be expressed exactly (see § 37), but we can express their values approximately. Thus, the table gives $\sqrt{2}=1.41421^+$ so that the two solutions become $4+1.41421^+$ and $4-1.41421^+$, which reduce to 5.41421^+ and 2.58579^+. *Ans.*

EXAMPLE 3. Solve $3\,x^2+8\,x=15$.

SOLUTION. Dividing through by 3 so as to have $+1$ as the coefficient of x^2, the equation becomes

$$x^2+\tfrac{8}{3}x=5.$$

Completing the square by adding $(\frac{4}{3})^2$ (or $\frac{16}{9}$) to both sides gives

$$x^2+\tfrac{8}{3}\,x+(\tfrac{4}{3})^2=5+\tfrac{16}{9}=\tfrac{61}{9},$$

$$(x+\tfrac{4}{3})^2=\tfrac{61}{9}.$$

king the square root of both members,

$$x+\tfrac{4}{3}=\pm\sqrt{\tfrac{61}{9}}=\pm\tfrac{1}{3}\sqrt{61}.$$

efore, the two solutions are

$$x=-\tfrac{4}{3}+\tfrac{1}{3}\sqrt{61} \text{ and } x=-\tfrac{4}{3}-\tfrac{1}{3}\sqrt{61}.$$

two answers may be written together in the condensed $\frac{1}{3}(-4\pm\sqrt{61})$ and by looking up the value of $\sqrt{61}$ in the ese values of x may be determined approximately, as in the Note to Example 2.

54. Summary and Rule. It is now to be observed carefully that in each of the three examples just considered (§ 53) the first step in the solution consists in reducing the given equation to the type form

$$x^2+px=q,$$

where p and q are given numbers.

Thus, in Example 3, we first put the equation $3x^2+8x=15$ into the form $x^2+\frac{8}{3}x=5$. Here $p=\frac{8}{3}$, and $q=5$.

The next step is to complete the square. This is done in each case by adding to both members the square of half the coefficient of x, that is we add $(p/2)^2$ to both members.

Thus, in Example 3, we had $p=\frac{8}{3}$, so we added $(\frac{4}{3})^2$ to both members.

After this, the equation is such that we can extract the square root of the left member, and when we do so and equate results, we obtain two simple equations, each yielding a solution of the given quadratic.

This may now be summarized in the following rule.

To solve any quadratic:

1. *Reduce the equation to the form*

$$x^2+px=q.$$

2. *Complete the square by adding $(p/2)^2$ to both members.*

3. *Extract the square root of both members of the new equation and equate results. This yields the two solutions desired.*

EXERCISES

Solve each of the following equations, checking your answers in the first five.

1. $x^2-5x=14$.
2. $x^2-20x=21$.
3. $x^2-12x+20=0$.
4. $x^2-2x=11$.
5. $x^2-3x-5=0$.
6. $8x=x^2-180$.
7. $x^2+22x=-12[illegible]$
8. $y^2=10-3y$.
9. $x^2-11x+28=[illegible]$
10. $6x^2-5x-6[illegible]$

11. $2x^2+5x+2=0.$

12. $1-3x=2x^2.$

13. $2x(x+4)=42.$

14. $(3x-2)^2=6x+11.$

15. $x+\frac{15}{x}=16.$

16. $\frac{9x^2}{4}-3x=-1.$

17. $\frac{x^2-24}{x}=10.$

18. $\frac{72}{x-6}=x.$

19. $\frac{x-3}{x+5}+\frac{x-6}{x-2}=0.$

55. Solution of Quadratics by the Hindu Method. A simple way preferred by many for completing the square in any quadratic is the one called the *Hindu method.* It consists of two steps:

1. *Multiply both members by four times the coefficient of* x^2.

2. *Add to both members of the new equation the square of the original coefficient of* x.

EXAMPLE. Solve $2x^2-3x=2.$

SOLUTION. Multiplying through by 4 times the coefficient of x^2, that is by 8, gives $16x^2-24x=16.$

Adding the square of the original coefficient of x to both sides, hat is adding $(-3)^2$, or 9, to both sides, gives

$$16x^2-24x+9=25.$$

The first member is now a perfect square, being equal to $(4x-3)^2$. refore, extracting square roots, we obtain

$$4x-3=5 \text{ and } 4x-3=-5.$$

lving the last two equations gives $x=2$ and $x=-\frac{1}{2}$. *Ans.*

EXERCISES

e each of the following quadratics by any method.

$+6x=35.$

$-12x=27.$

$-x-3=0.$

4. $16x^2-7x-123=0.$

5. $16x^2+8x=1.$

6. $25x^2-9x=16.$

7. $2x^2+6x=\frac{7}{2}$.
[HINT. First multiply both numbers by 2.]

8. $3x^2-2x=5$.

9. $3x^2+7x-110=0$.

10. $5x^2-7x=-2$.

11. $1-3x=2x^2$.

12. $4x^2-3x-2=0$.

13. $2x^2+3x=27$.

14. $3x^2-7x+2=0$.

15. $4x^2-17x=-4$.

16. $8x=x^2-180$.

17. $3x^2+x-200=0$.

18. $x+\frac{1}{x}=\frac{5}{2}$.

19. $\frac{x}{x-5}-\frac{x-5}{x}=\frac{3}{2}$.

20. $\frac{x}{x+2}+\frac{1}{2}=\frac{x+2}{2x}$.

21. $x+\sqrt{2x+3}=6$.
[HINT. Proceed as in § 40.]

22. $\sqrt{x+1}-\sqrt{x-2}=\sqrt{2x-5}$.

23. $\sqrt{x-1}+\sqrt{10-x}=3$.

56. Solution by Formula. Every quadratic is an equation of the type form

$$ax^2+bx+c=0,$$

where a, b, and c are given numbers. We may solve this equation as it stands by the process of § 54. Thus,

$$ax^2+bx=-c.$$

Dividing through by a,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Adding $b/(2a)^2$ to both sides (§ 54) gives

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

Extracting the square root of both members,

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}.$$

Transposing the term $b/(2a)$, we thus have the fol[illegible] formulas for the two roots:

$$\boldsymbol{x=\frac{-b+\sqrt{b^2-4ac}}{2a}} \quad \text{and} \quad \boldsymbol{x=\frac{-b-\sqrt{b^2-4ac}}{2a}}$$

NOTE. Observe that these formulas express the values of x *in terms of* the known letters a, b, and c, as should be the case. Thus, in any example, we have only to put the given values of a, b, and c into the formulas in order to have *at once* the desired values of the two roots.

EXAMPLE. Solve by formula $2x^2-3x=2$.

SOLUTION. This equation may be written $2x^2-3x-2=0$. Hence the values of a, b, and c in this case are as follows: $a=2$, $b=-3$, $c=-2$. Placing these values in the formulas gives as the roots

$$x=\frac{-(-3)+\sqrt{(-3)^2-4(2)(-2)}}{2\cdot 2},$$

and
$$x=\frac{-(-3)-\sqrt{(-3)^2-4(2)(-2)}}{2\cdot 2}.$$

Simplifying, $x=\dfrac{3+\sqrt{9+16}}{4}=\dfrac{3+\sqrt{25}}{4}=\dfrac{3+5}{4}=2,$

and
$$x=\frac{3-\sqrt{9+16}}{4}=\frac{3-5}{4}=-\frac{1}{2}.$$

The two roots are, therefore, $x=2$ and $x=-\frac{1}{2}$. *Ans.* (Compare solution in § 55.)

EXERCISES

1. Solve *by the formula* Exs. 13–17, p. 87.

2. Solve by the formula the equation $3x^2-6x+2=0$.

[HINT. The roots are found to be $x=1+\frac{1}{3}\sqrt{3}$ and $x=1-\frac{1}{3}\sqrt{3}$. his quadratic thus has roots which necessarily contain radicals. m the table of square roots at the end of the book, we find $=1.73205$. Hence, the roots, correct to five decimal places, are $\frac{1.73205}{3}$ and $1-\frac{1.73205}{3}$, which reduce respectively to 1.57735 42265. *Ans.*]

Solve by the formula the equation $x^2-5x+3=0$, and tables if necessary to determine the roots decimally.

olve by the formula the equation $4x^2-3x-2=0$, ex- the roots decimally correct to five places.

APPLIED PROBLEMS

[The method of solving quadratics by formula (§ 56) is usually the most direct.]

1. The square of a certain number is 4 less than five times the number. Find the number.

[HINT. Remember that there should be two solutions.]

2. Divide 20 into two parts whose product is 96.

3. One side of a right triangle is 2 inches longer than the other. If the hypotenuse is 10 inches long, how long are the sides?

[HINT. Letting x represent the shorter side, the equation here becomes $x^2+(x+2)^2=100$, and in solving this we find that one of the solutions is *negative*. But a negative solution can have no meaning in such an example as this, so we keep only the positive solution. This frequently happens in applied problems involving quadratics, so the pupil must always be on his guard to keep only such solutions as can actually fit a given problem.]

4. A gardener spades a plot 30 feet long by 20 feet wide. He then decides to double its area by adding a border of uniform width throughout. How wide must the border be made?

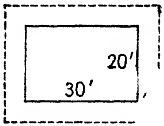


FIG. 13.

5. In Example 4 suppose that instead of doubling the area, the gardener wishes merely to add 200 square feet to it. Show that the strip added around the outside must then be made a lit over 1.86 feet wide.

6. A circular swimming pool is surrounded by a wa feet wide. If the area of the walk is one fourth that pool, find (approximately) the radius of the pool. $\pi=3\frac{1}{7}$.)

7. If a train had its speed diminished by 10 miles an hour, it would take it 1 hour longer to travel 200 miles. What is the speed?

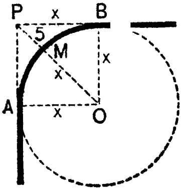


Fig. 14.

8. A circular curbing touches the line of the street curbing on each of two streets that meet at right angles. In Fig. 14, the middle point of the circular curbing is marked M. The point at which the straight curbings would meet if extended is marked P. If $PM = 5$ ft., find the radius (x in Fig. 14) of the circular curbing.†

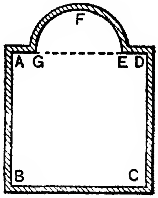


Fig. 15.

9. The figure represents a pattern frequently used in window designs, consisting of a square $ABCD$ with a semicircle EFG mounted upon it, the diameter GE of the semicircle being slightly less than one of the sides of the square. If the shoulders AG and DE are each 1 foot long, how i ust each side of the square be made r that the total lighting surface shall be 88 square f

10. A soap bubble of rad s r is blown out until the area of its outer surface becomes double its original value. Show at the radius has thus bee increased by an amount h given the formula $h = r(\sqrt{2} - 1)$. [Hint. See Ex. 14 (*f*), p. 6.]

7. **Graphical Solution of Quadratics.** Consider the dratic $x^2 - 3x - 4 = 0$. Let us represent the left member he letter y; that is, let us place

$$y = x^2 - 3x - 4.$$

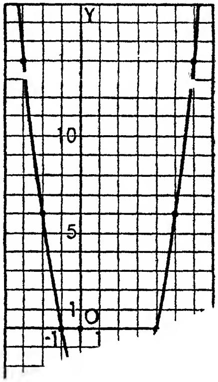
, if we give to x any value, this equation determines a

s problem suggests a practical plan for finding the radius ar curbings when the center O of the circle cannot be reached.

corresponding value for y. For example, if $x=0$, then $y=0^2-3\times 0-4=-4$. Again, if $x=1$, then $y=1^2-3\times 1-4=-6$. The table below shows a number of x-values with their corresponding y-values, determined in this way.

When $x =$	0	1	2	3	4	5	6	-1	-2	-3
then $y =$	-4	-6	-6	-4	0	6	14	0	6	14

The graph is now obtained by drawing an x-axis and a y-axis, as in § 28, then plotting each of the points x, y which the table contains, and finally drawing the smooth curve passing through all such points. The form of the graph thus obtained is indicated in the adjoining figure. Observe that this graph is *not* a straight line and is therefore different in character from the graph of a linear equation. (See § 29.) And it is especially important to notice that it cuts the x-axis in *two* points who… x-values are -1 and 4, res… These two values of … in this purely graphical … two solutions of the given qu… are those values of x tha… $x^2-3x-4=0$.



The graphical stud… special quadratic $x^2-3x-4=0$ leads … more general statements.

Every quadratic has a graph which is obt… y equal to the left member of the equation … that the right member is 0), then letting x t…

and determining their corresponding y-values, plotting the points, x, y, thus obtained and finally drawing the smooth curve through them.

The x-values of the two points where the graph cuts the x-axis will be the roots of the given quadratic.

EXERCISES

Draw the graphs of each of the following quadratics, and note where each cuts the x-axis. In this way determine graphically the solutions, and check the correctness of your answer by actually solving by one of the methods explained in §§ 54–56.

1. $x^2-x-2=0$. **2.** $x^2-7x+12=0$. **3.** $x^2+7x+12=0$.

4. $x^2-5x=-6$.

[HINT. Remember to write first as $x^2-5x+6=0$.]

5. $x^2+3x=10$. **6.** $2x^2+3x=9$.

Having Imaginary Solutions. Consider
This is a pure quadratic (§ 49) and
tely by merely taking the
gives as the required solu-
-1. But $\sqrt{-1}$ means the
nd *there is no such number*
) *which we have thus far met.*
ny number, whether the
neg *sitive* [§ 2(*d*)]. There-
e as this, we say that the solutions are
speak of the numbers themselves which,
into algebra in this way, as ***imaginary***
re *imaginary*, however, only in the sense
been encountered before.

As an example of an affected quadratic having imaginary roots, let us consider the equation $x^2-6x+15=0$. When we proceed to solve this by the method of completing the square, as in § 54, the work is as follows.

Transposing, we have

$$x^2-6x=-15.$$

Adding 9 to both sides to complete the square,

$$x^2-6x+9=-6, \text{ or } (x-3)^2=-6.$$

Extracting the square root of both sides,

$$x-3=\pm\sqrt{-6}.$$

Therefore the solutions are

$$x=3+\sqrt{-6} \text{ and } x=3-\sqrt{-6}. \quad \textit{Ans.}$$

Both of these solutions are seen to be imaginary because they contain the expression $\sqrt{-6}$.

59. Definitions. A number like $3+\sqrt{-6}$ or $3-\sqrt{-6}$ is frequently called a ***complex number*** in distinction to such a number as $\sqrt{-6}$, which is called a ***pure imaginary***. Thus, a complex number is a combination of a positive or negative number with a pure imaginary.

All numbers considered in the chapters preceding this (including irrationals) are called ***real*** numbers in distinction from the imaginary numbers just described. Thus, the sol tions of all quadratics considered in §§ 54–56 are real inst of imaginary.

EXERCISES

Find (by solving) whether the solutions of the follo quadratics are real or imaginary.

1. $x^2+9=0$. **3.** $2x^2+2x+3=0$. **5.** $x^2=-4(x$

2. $x^2-6x+10=0$. **4.** $3x^2+2x=4$. **6.** $4x^2-3x$

60. Determining Graphically Whether Solutions are Real or Imaginary. It was shown in § 58 that the solutions of the quadratic $x^2-6x=-15$ are imaginary. Let us now see what corresponds to this fact in the graph.

The table below shows several values of x and their corresponding y-values, as determined from the given equation

$$y=x^2-6x+15.$$

When $x=$	-1	0	1	2	3	4	5	6
then $y=$	22	15	10	7	6	7	10	15

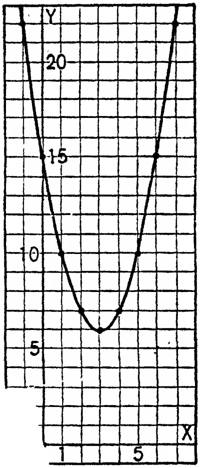


FIG. 17.

Plotting the various points (x, y) thus obtained and drawing the curve through them gives the graph indicated in the accompanying figure. This graph is essentially different from those met with in § 57 in one particular, namely *it does not cut the x-axis.*

A similar result holds for the graph of every quadratic whose solutions are imaginary. Therefore, in order to tell whether the solutions of any given quadratic are real or imaginary, we need only draw its graph and note whether or not it cuts the x-axis. If it does, the solutions are real; if it does not, the solutions are imaginary.

EXERCISES

ıd by drawing the graph whether the roots of each of the ing quadratics are real or imaginary.

$2x+3=0.$ **3.** $x^2-2x+3=0.$ **5.** $6x^2+5x+1=0.$

$-2x-3=0.$ **4.** $3x^2+4x+1=0.$ **6.** $2x^2-3x+4=0.$

LAGRANGE

(*Joseph Louis Lagrange*, 1736–1813)

Famous for his discoveries in all branches of mathematics and regarded as the greatest mathematician of the 18th century. In algebra he gave much attention to the study of equations and determinants, extending and unifying the work of previous mathematicians in these fields.

PART II. ADVANCED TOPICS

CHAPTER XI

LITERAL EQUATIONS AND FORMULAS

61. Literal Equations. Equations in which some, or all, of the known numbers are represented by letters are called ***literal equations.*** The known letters are generally represented by the first letters of the alphabet, as a, b, c, etc. Literal equations are solved by the same processes as numerical equations.

EXAMPLE. Solve the following literal equation for x:

$$ax = bx + 7c.$$

SOLUTION. Transposing,

$$ax - bx = 7c.$$

Combining like terms,

$$(a-b)x = 7c.$$

Dividing by $(a-b)$,

$$x = \frac{7c}{a-b}. \quad \textit{Ans.}$$

CHECK. Substituting the answer for x in the given equation,

$$a\left(\frac{7c}{a-b}\right) = b\left(\frac{7c}{a-b}\right) + 7c.$$

Multiplying by $(a-b)$,

$$7ac = 7bc + 7c(a-b) = 7bc + 7ac - 7bc.$$

Transposing,

$$7ac - 7ac = 7bc - 7bc.$$

Simplifying, $\qquad 0 = 0$, which is a correct result.

It is to be carefully observed that a literal equation is said to be solved for the unknown letter, as x, only when that letter has been expressed *in terms of* the other (known) letters. Thus, in the example above, we obtained x in terms of a, b, and c. This when once done, is what we mean by the solution.

NOTE. If a literal equation is satisfied no matter what values be given to the letters appearing in it, it is called an ***identity***. Thus $x^2-a^2=(x-a)(x+a)$ is an identity. This fact is often expressed by means of the symbol $\equiv$. Thus, $x^2-a^2\equiv(x-a)(x+a)$. Likewise, $(x+a)^2\equiv x^2+2\,ax+a^2$, $(x-a)^2\equiv x^2-2\,ax+a^2$, etc.

EXERCISES

Solve for x in the following, checking your answer in the first five.

1. $x-a=b$.
2. $ax-1=b$.
3. $ax+bx=c$.
4. $3\,x+b=x-3\,b$.
5. $4(3\,b-x)=3(2\,b+x)$.
6. $(x-a)(x-b)=x(x+c)$.
7. $\dfrac{1}{1+x}=a$.
8. $\dfrac{a}{x}+\dfrac{b}{x}=2$.
9. $\dfrac{x}{a}+b=\dfrac{x}{b}+a$.
10. $\dfrac{a}{cx}+\dfrac{b}{dx}=e$.
11. $\dfrac{x-c}{c}+a=x-1$.
12. $\dfrac{x-b}{x-3}+\dfrac{x-c}{x+2}=2$.
13. Divide a into two parts whose quotient is m.

14. If A can do a piece of work in a days, and B can do it in b days, how long will it take them working together? (See Exs. 23, 24, p. 40.)

15. $$\begin{aligned}3\,x+5\,y&=2\,a,\\ 2\,x-3\,y&=4\,b.\end{aligned}$$

[HINT. These are simultaneous equations, to be solved for the two unknowns x and y *in terms of a and b*.]

16. $\begin{cases} ax-by=2, \\ cx+dy=3. \end{cases}$ **17.** $\begin{cases} 3\,ax+2\,by=ab, \\ ax-by=ab. \end{cases}$

18. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a}, \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{b}. \end{cases}$

[HINT. Solve first for $1/x$ and $1/y$. See Ex. 16, p. 53.]

19. $\begin{cases} \dfrac{a}{x}-\dfrac{b}{y}=-1, \\ \dfrac{b}{x}-\dfrac{a}{y}=-1. \end{cases}$

20. $ax^2-c=1.$

[HINT. See § 51. *Ans.* $x=\pm\sqrt{\dfrac{c+1}{a}}.$]

21. $ax^2+a^3=5\,a^3-3\,ax^2.$

22. $\dfrac{a}{x}+\dfrac{x}{a}=\dfrac{1}{x}.$

23. $\dfrac{x}{a+b}-\dfrac{a-b}{x}=0.$

24. $(x+a)(x+b)+4(x+a)=0.$

[HINT. Solve by factoring.]

25. $x^2-ax=2\,a^2.$

[HINT. See § 54, p. 85. *Ans.* $x=2\,a$, or $x=-a$.]

26. $4\,x^2=7\,m^2-12\,mx.$

27. $x^2-(a-b)\,x=ab.$

28. $x^2+ax=ac+cx.$

29. $x^2=4\,ax-2\,a^2.$

30. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

62. Formulas. If a person travels for 10 hours at the rate of 15 miles an hour, the distance he travels is $15 \times 10 = 150$ miles. Stated in general (algebraic) language, we can say in the same way that if a person travels for t hours at the rate of r miles an hour, the distance s he travels is

$$s = rt.$$

This is a literal equation expressing the value of s in terms of r and t. If we wish, we can solve it for t, giving $t = s/r$, and what we now have is t expressed in terms of s and r. Or, we can solve the original equation for r, giving $r = s/t$, and this expresses r in terms of s and t.

These examples illustrate the important fact that in nearly all branches of knowledge, especially in engineering, geometry, physics, and the like, there are *general* laws which are expressed by means of mathematical formulas. Such formulas are merely literal equations in which two or more letters appear, and it is often desirable to solve them for some one letter in order to express its value in terms of the others.

EXERCISES

1. The area A of a rectangle whose dimensions (length and breadth) are a and b is given by the formula $A = ab$. Solve this for a; also for b. In each case state in terms of what letters your answer is written.

2. The formula for the area A of a triangle whose height (altitude) is h and whose base is a is $A = \frac{1}{2}ah$. Solve for a; also for h.

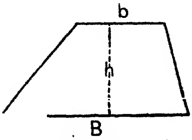


FIG. 18.

3. Solve for b in the formula

$$A = \tfrac{1}{2}h(B + b).$$

(Formula for the area A of a trapezoid whose bases are B and b and whose altitude is h.)

4. Solve for r in the formula $A = \pi r^2$. (Formula for the area of a circle whose radius is r.)

5. The interest I which a principal of p dollars will yield in t years at $r\%$ is determined by the formula

$$I = \frac{prt}{100}.$$

Solve this for r and use your result to answer the following question: What rate of interest is necessary in order that \$50 may yield \$6 interest in 2 years' time?

[HINT. Solving for r gives at once

$$r = \frac{100\,I}{pt}.$$

Now see what the right member of this equation becomes when $I = 6$, $p = 50$, and $t = 2$.]

6. Using the interest formula of Ex. 5, solve it for t and use your result to answer the following question: How long will it take \$600 to yield \$63 interest if invested at 6%?

7. The velocity of sound v, in feet per second, is given by the formula $v = 1090 + 1.14(t - 32)$, where t is the temperature of the air in Fahrenheit degrees. Find

(*a*) The velocity of sound when the temperature is 75°.

(*b*) The temperature when sound travels 1120 ft. per sec.

8. Derive formulas for each of the following statements.

(*a*) The number N of turns made by a wagon wheel d feet in diameter in traveling s miles.

(*b*) The number N of dimes in m dollars, n quarter dollars, and q cents.

9. An automobile travels for T hours at the rate of v miles per hour. By how much must this rate be increased in order to make the same journey in t minutes less time?

10. A has \$$a$ and B has \$$b$. Between them they give \$$c$ to a certain charity, after which the amounts of money they have are equal. How much does each contribute?

63. Law of the Lever. If two weights are balanced at the ends of any (uniform) bar, as shown in the figure, we have an example of a *lever*. The point of support, F, is called the *fulcrum*. If we let W and w be the values (in pounds or ounces or any other convenient unit) of the two weights, while D and d stand for the distances respectively of W and w from F, then, whenever the balance is perfect, we have the formula

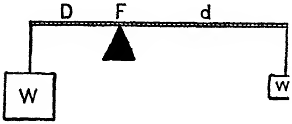


FIG. 19.

$$\frac{W}{w} = \frac{d}{D}.$$

Sometimes a single weight W is balanced by a force, p, usually called a *power*. This may happen in several ways, as indicated by the following figures. In all such cases, if we

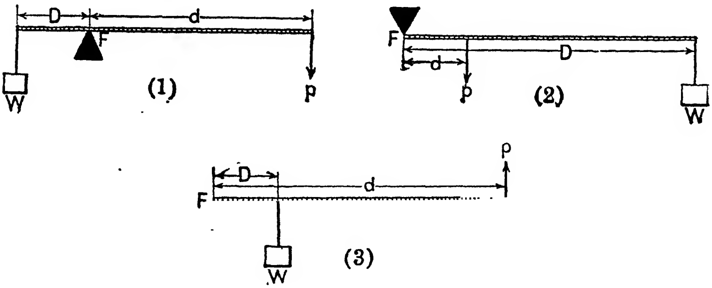


FIG. 20.

let W represent the weight, p the power, D the distance from W to the fulcrum, there exists the following formula whenever the balance is perfect:

$$\frac{W}{p} = \frac{d}{D}.$$

This is called *the general law of the lever.* By clearing the equation of fractions, it may be written in the form

$$WD = pd.$$

Translated into words, this last relation means that *the weight times the weight arm equals the power times the power arm.* It is in this form that the law is usually remembered by engineers.

EXERCISES ON THE LEVER

1. If the fulcrum of a 5-foot crowbar is placed 1 foot from the end, what weight can be lifted by a man weighing 180 pounds?

[HINT. Here we have Fig. 19 with $W = ?$, w (or p) $= 180$ pounds, $D = 1$ foot, $d = 4$ feet.]

2. The figure represents a simple form of pump. If the pump handle AF is 16 inches long, while the piston-arm FC is 3 inches long, what will be the upward pull at C when there is a 9-pound downward push at A?

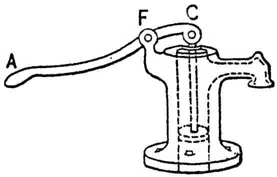


FIG. 21.

3. A certain lever, after being balanced, has r pounds added to the weight W. Determine (in terms of $W, p, D, d,$ and r) how much the power, p, must be increased to keep the balance perfect.

64. Gear Wheel Law. Whenever a gear wheel having T teeth revolves at the rate of N revolutions per minute and turns another similar wheel having t teeth at the rate of n revolutions per minute, there exists at all times during the motion the formula

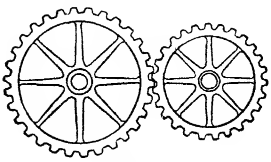


FIG. 22.

$$\frac{T}{t} = \frac{n}{N}.$$

This is the *gear wheel law.* Clearing this equation of fractions, it becomes

$$TN = tn.$$

Translated into words, this last relation means that the *number of teeth in one wheel multiplied by its rate of turning is equal to the number of teeth in the other wheel multiplied by its rate of turning.*

EXERCISES ON GEAR WHEELS

1. If in Fig. 22 the small wheel has 30 teeth and is making 96 revolutions per minute, how many teeth must the large wheel have in order to revolve 16 times per minute?

2. In order that the large wheel revolve three fourths as fast as the small one, how must the wheels be made?

3. If the large wheel in § 64 be made larger by the addition of r teeth to its rim, determine the amount by which the speed of the smaller wheel will be thereby increased.

[HINT. Let x represent the unknown amount and find a formula for x in terms of T, t, N, n, and r.]

65. Other Useful Formulas. In addition to the formulas already mentioned, the following from plane and solid geometry and from elementary physics are often used.

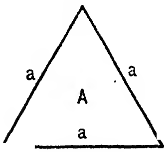


FIG. 23.

1. The area A of an equilateral triangle of side a (Fig. 23) is

$$A = \frac{\sqrt{3}}{4} a^2,$$

where $\sqrt{3} = 1.732$ approximately.

2. The area A of a regular hexagon of side a (Fig. 24) is

$$A = \frac{3\sqrt{3}}{2} a^2.$$

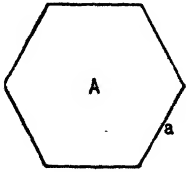


FIG. 24.

3. The area A of any triangle in terms of its three sides a, b, and c (Fig. 25) is

$$A=\sqrt{s(s-a)(s-b)(s-c)},$$

where $s=\frac{1}{2}(a+b+c)$.

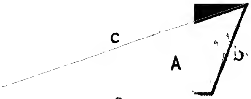


FIG. 25.

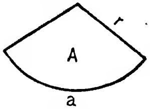


FIG. 26.

4. The area A of the sector of a circle when the intercepted arc is a and the radius is r (Fig. 26), is

$$A=\frac{1}{2}ar.$$

5. The length of the diagonal d of a rectangular block whose dimensions are a, b, and c (Fig. 27) is

$$d=\sqrt{a^2+b^2+c^2}.$$

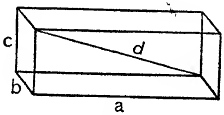


FIG 27.

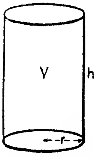


FIG. 28.

6. The volume V of a circular cylinder of altitude h and radius of base r (Fig. 28) is

$$V=\pi r^2 h.$$

7. The volume V of a circular cone of altitude h and radius of base r (see Fig. 29) is

$$V=\frac{1}{3}\pi r^2 h.$$

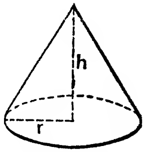


FIG. 29.

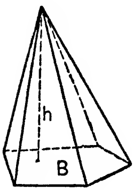


FIG. 30.

8. The volume V of a pyramid of altitude h and base B (Fig. 30) is

$$V=\frac{1}{3}Bh.$$

9. The volume V of a spherical segment (or slice of a sphere between two parallel cutting planes), where h is the altitude, and a and b the radii of the two bases (Fig. 31) is

$$V=\frac{\pi h}{2}\left[(a^2+b^2)+\frac{h^2}{3}\right].$$

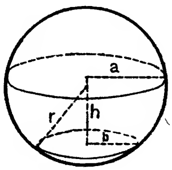


FIG. 31.

10. The surface S of the zone (or portion of the surface of a sphere lying between two parallel cutting planes), where h is the distance between the cutting planes and r the radius of the sphere (Fig. 31), is

$$S = 2\,\pi r h.$$

11. The length of belt l required to go around two wheels whose diameters are D and d and whose centers are at the distance a apart is

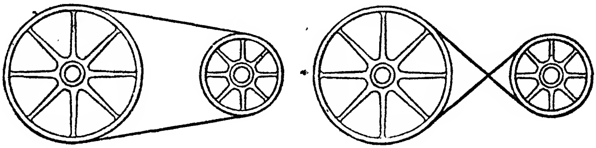


Fig. 32.

(a) In case the belt does not cross itself,

$$l = 2\sqrt{\left(\frac{D-d}{2}\right)^2 + a^2} + \pi\frac{D+d}{2}.$$

(b) In case the belt crosses itself once,

$$l = 2\sqrt{\left(\frac{D+d}{2}\right)^2 + a^2} + \pi\frac{D+d}{2}.$$

12. The force F, measured in pounds, with which a body weighing W pounds pulls outward (centrifugal force) when traveling with a velocity of v feet per second in a circle of radius r (Fig. 33) is determined by the formula

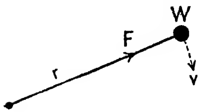


Fig. 33.

$$F = \frac{Wv^2}{32\,r}.$$

13. The pressure P exerted by a letter press (Fig. 34) is determined by the formula

$$P = \frac{2\,\pi r F}{h},$$

where F is the value of the force applied at

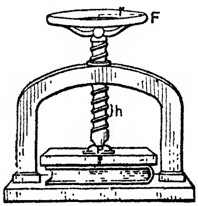


Fig. 34.

the wheel, r is the radius of the wheel, and h is the distance from one thread of the screw to the next one.

14. The weight W which can be raised by means of a toothed wheel and screw such as indicated in Fig. 35 is determined by the formula

$$W = \frac{2\pi lRP}{dr},$$

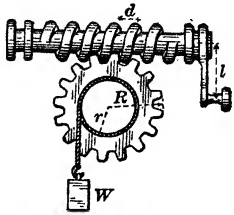


FIG. 35.

where P represents the pressure applied at the handle and where R, r, d, and l are the dimensions indicated in the figure.

Other important formulas from elementary geometry and from physics may be found in Chapter XVII.

EXERCISES

1. Find by Formula 3 of § 65 the area of the triangle whose three sides are respectively 5 inches, 5 inches, and 8 inches long.

2. Show that in the case of an equilateral triangle of side a the expression for A in Formula 3, § 65, reduces, as it should, to that for A in Formula 1, § 65.

3. Show that in case the two wheels in Fig. 32 have the *same* diameter D, the formula for the length of belt reduces in case (a) to the simple form $l = \pi D + 2a$, and in case (b) to $l = 2\sqrt{D^2 + a^2} + \pi D$.

4. How much leather (surface measure) will it take for a belt 6 inches wide to connect two pulleys whose diameters are 5 feet and 1 foot, respectively, the distance between centers being 10 feet?

[HINT. $\sqrt{104} = 10.2$, approximately.]

5. A pail of water weighing 5 pounds is swung round at arm's length at a speed of 10 feet per second. If the length of the arm is 2 feet, find (*a*) the pull at the shoulder when the pail is at the uppermost point of its course, (*b*) when at the lowest point of its course. Also, find the least velocity which the pail can have without the water dropping out at the top point of the course.

6. What pressure is exerted by a letter press in case the force applied at the wheel is 10 pounds, the diameter of the wheel is $1\frac{1}{2}$ feet, and the threads of the screw are $\frac{1}{2}$ inch apart?

7. In the device shown in Fig. 35, show that if the distance d between two adjacent threads be halved and the number of teeth on the wheel be correspondingly doubled to fit the new gear, other parts remaining the same, the weight W that can be raised with a given pressure P will be doubled.

8. The volume V of the frustum of a cone or pyramid made by a plane parallel to the base is given by the formula

$$V=\frac{h}{3}(B+b+\sqrt{Bb}),$$

where B and b denote the areas of the lower and upper bases, and h denotes the altitude.

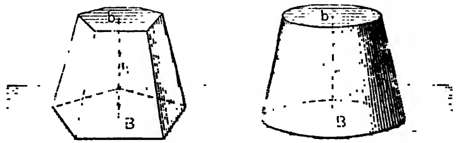


FIG. 36.

Determine the volume of the frustum of a circular cone the radii of whose bases are 4 inches and 3 inches, the altitude being 5 inches.

Show that in case the upper and lower bases are equilateral triangles whose sides are a and b respectively, the formula becomes

$$V=\frac{\sqrt{3}}{12}h(a^2+ab+b^2).$$

[HINT. See Formula 1, § 65.]

9. Show by means of Formula 3, p. 103, that if the three sides of any triangle be extended to twice their original length, the area will become quadrupled.

10. If, in the first of the two cases represented in Fig. 32, the diameter of each wheel be increased by the same amount, say a, show that the length of belt will thereby become increased by the amount πa.

11. If, in the second of the cases represented in Fig. 32, the diameter of the large wheel be increased by any given amount and at the same time the diameter of the small wheel be diminished by the same amount, show that precisely the same length of belt as before will fit over them tightly.

12. A body is moving along a circle with a velocity of 3 feet per second. Show that in order for the centrifugal force (12, p. 104) which it is exerting to be doubled, the velocity must be increased by about $1\frac{1}{4}$ feet per second.

By how much would the velocity have to be *diminished* in order that the centrifugal force become halved?

13. By means of Formulas 9 and 10 (§ 65) obtain the formulas for the area and volume of a whole sphere. Compare with Ex. 14, (*e*), (*f*), p. 6.

CHAPTER XII

GENERAL PROPERTIES OF QUADRATIC EQUATIONS

66. The Classification of Numbers. A ***real number*** is one whose expression does not require the square root of a negative quantity, while an ***imaginary number*** is one whose expression does require such a square root. (See § 59, p. 93.)

Thus $1, 3, -7, \frac{1}{2}, \frac{2}{3}, -\frac{8}{9}, \sqrt{2}, 1+\sqrt{3}$ are real numbers, while $\sqrt{-3}, \sqrt{-\frac{1}{2}}, 2+\sqrt{-3}$, are imaginary numbers.

In case a real number can be expressed in the particular form $\frac{p}{q}$ where p and q are integers (positive or negative) it is called a ***rational number.*** The number zero is also included among the rational numbers. (See § 42, p. 56.)

Thus $\frac{1}{2}, \frac{2}{3}, -\frac{4}{7}, 5, 73, -10, -\frac{17}{16}$, are rational numbers.

In case a real number cannot be expressed in the particular form just mentioned, it is called an ***irrational number.***

Thus $\sqrt{2}, \sqrt{3}, \sqrt{\frac{2}{3}}, \sqrt[3]{2}, \sqrt[3]{\frac{1}{2}}, \sqrt[3]{9}, 1+\sqrt{6}$, are irrational numbers.

Imaginary numbers are either ***pure imaginaries***, such as $\sqrt{-3}$, or ***complex numbers*** (§ 59, p. 93), such as $1+\sqrt{-3}$.

These divisions and subdivisions may be summarized into a table as below:

$$\textit{Numbers of Algebra}\begin{cases}\textit{Real}\begin{cases}\textit{Rational}\\ \textit{Irrational}\end{cases}\\ \textit{Imaginary}\begin{cases}\textit{Pure Imaginaries}\\ \textit{Complex Numbers}\end{cases}\end{cases}$$

NOTE. The rational numbers are themselves subdivided into three sub-classes: the single number *zero*, the *integers* (positive or negative), and the *rational fractions*. The latter are those rational numbers which, like $\frac{2}{3}$, cannot be expressed as integers.

All the numbers used in arithmetic are positive real numbers. The negative real numbers together with the imaginary numbers owe their existence to algebra.

67. Determining the Character of the Roots of a Quadratic. It is often desirable to determine the character of the roots of a given quadratic, that is, whether the roots are real or imaginary; and if real, whether they are rational or irrational, etc.

Thus the roots of $2x^2-7x+1=0$ are (by § 56, p. 87), $\dfrac{7\pm\sqrt{41}}{4}$.

Since 41 is positive, these roots are real numbers (§ 66).

Since 41 is not a perfect square, the roots are irrational (§ 66).

Since $\sqrt{41}$ is added to 7 in the one root and subtracted from 7 in the other root, the roots are unequal.

Thus the character of the roots in this case is described by saying that they are real, irrational and unequal.

There is, however, a much shorter method than the one illustrated above for determining the character of the roots of a given quadratic. Thus we know (§ 56) that the two roots of any quadratic, namely, any equation of the form

$$ax^2+bx+c=0,$$

are $$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad\text{and}\quad \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

An examination of the form of these expressions gives at once the following rule.

RULE. *For any given quadratic, $ax^2+bx+c=0$, in which the coefficients a, b, c are real numbers, the two roots will be*

(1) *Real and unequal if b^2-4ac is positive.*

(2) *Real and equal if $b^2-4ac=0$.* (Both roots then reduce to $-b/2a$.)

(3) *Imaginary if b^2-4ac is negative.*

Moreover, if the coefficients a, b, c are rational numbers, the two roots will be

(4) *Rational, if b^2-4ac is a perfect square; irrational if b^2-4ac is not a perfect square.*

Because of the manner in which the character of the roots thus comes to depend upon the value of b^2-4ac, this expression is called the ***discriminant*** of the given quadratic.

EXAMPLE 1. Determine the character of the roots of $2x^2-3x-9=0$.

SOLUTION. Here $a=2$, $b=-3$, $c=-9$. Hence the value of the discriminant, or b^2-4ac, is $(-3)^2-4(2)(-9)=9+72=81=9^2$.

Hence, by (1) and (4) of the rule, the roots are real, unequal, and rational.

EXAMPLE 2. Determine the character of the roots of $3x^2+2x+1=0$.

SOLUTION. Here $a=3$, $b=2$, $c=1$. Hence $b^2-4ac=4-12=-8$. Hence, by (3) of the rule, the two roots must be imaginary.

EXAMPLE 3. Determine the character of the roots of $4x^2-20x+25=0$.

SOLUTION. $a=4$, $b=-20$, $c=25$. Hence $b^2-4ac=400-400=0$. Therefore, by (2) of the rule, the roots are real and equal.

The common value which the two roots have may be found if desired by actually solving the equation. It turns out to be $\frac{5}{2}$.

EXERCISES

Determine (without solving) the character of the roots of each of the following equations.

1. $2x^2-3x+1=0$.
2. $2x^2-4x+1=0$.
3. $x^2+5x+6=0$.
4. $3x^2-x-10=0$.
5. $3x^2-x+10=0$.
6. $x^2+x=1$.
7. $x^2+x=-1$.
8. $4x^2-4x+1=0$.
9. $\frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{5}=0$.
10. $7x^2+3x=0$.
11. $5r-2=4r^2$.
12. $5s^2+7=8s$.

***68. Character of Roots Considered Geometrically.** We have seen in § 57 that whenever a quadratic has its two roots real and unequal, its graph will cut the x-axis in two distinct points. On the other hand, if the roots are imaginary, the graph of the equation will not cut the x-axis at all (§ 60). Suppose now that we have a quadratic whose two roots are *equal* to each other, for example

$$(1)\qquad 4x^2-12x+9=0.$$

Here the discriminant is

$$(-12)^2-4(4)(9)=144-144=0,$$

so that the roots must be equal (§ 67).

If we now proceed to draw the graph of (1) in the usual way by placing $y=4x^2-12x+9$, then forming a table of x, y values, etc., it appears that the graph corresponding to (1) just *touches* the x-axis instead of cutting through it. This, in fact, is what we should expect, since the equality of the roots virtually means that there is but one root, and this can be possible only when the graph merely touches (is tangent to) the x-axis.

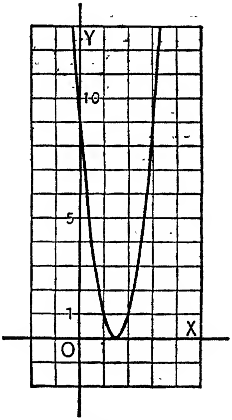


FIG. 37.

In order to illustrate in one single diagram all the facts mentioned thus far about the graph, it is instructive to take such an equation as

(2) $$x^2-2x+c=0,$$

and let c take different values, thus obtaining various quadratics. For example, if we choose $c=-2$, the quadratic equation becomes

$$x^2-2x-2=0;$$

and if we draw the graph of this, we find that it cuts through the x-axis at two points, thus indicating that its solutions are real and unequal.

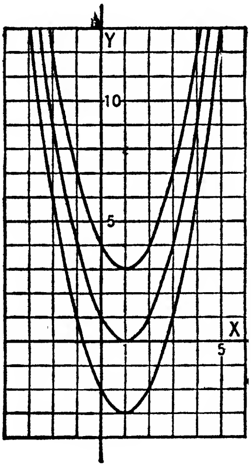


FIG. 38.

Likewise, we find a similar result when we let $c=-1$, and when $c=0$, though the various graphs are themselves different.

But if we choose $c=1$ and proceed as before, the graph of the corresponding quadratic no longer cuts through the x-axis, but merely touches it, thus indicating real and equal roots.

Finally, for such values of c as 2, 3, or 4, the quadratics (2) come to have graphs which do not cut the x-axis, thus indicating that they have imaginary roots. The effect of changing c is thus merely to slide one and the same curve vertically up and down the coördinate paper. The figure shows the positions of the curve corresponding to $c=-2$, $c=1$, and $c=4$. The pupil is advised to draw in for himself the positions of the curve for $c=-1$, $c=0$, $c=2$, and $c=3$.

69. The Sum and Product of the Roots. We have seen that every quadratic is an equation of the form

(1) $$ax^2+bx+c=0;$$

and we have also seen (§ 56) that the roots, or solutions, of this equation are

$$x_1=\frac{-b+\sqrt{b^2-4\,ac}}{2\,a},$$

and

$$x_2=\frac{-b-\sqrt{b^2-4\,ac}}{2\,a}.$$

It is now to be observed that if we add these solutions together the radical cancels and we obtain the simple result

$$x_1+x_2=\frac{-2\,b}{2\,a}=-\frac{b}{a}.$$

Again, if we multiply the two solutions together, we obtain a final result which is very simple in form. Thus

$$x_1\cdot x_2=\frac{(-b)^2-(\sqrt{b^2-4\,ac})^2}{4\,a^2}=\frac{b^2-(b^2-4\,ac)}{4\,a^2}=\frac{4\,ac}{4\,a^2}=\frac{c}{a}.$$

These results may be summarized in the following rule.

Rule. *In the general quadratic equation* $ax^2+bx+c=0$,

(1) *The sum of the two solutions is* $-b/a$.

(2) *The product of the two solutions is* c/a.

Example. Find the sum and the product of the solutions of the equation $3\,x^2-2\,x+6=0$.

Solution. Here $a=3$, $b=-2$, $c=6$.

Hence the sum of the solutions is $-\frac{-2}{3}$, or $\frac{2}{3}$, and the product of the solutions is $\frac{6}{3}=2$.

EXERCISES

State (by inspection) the sum and the product of the solutions of each of the following equations. Check your answer in Exs. 1, 2, 3, 4 by actually solving the equations and determining the sum and the product of the solutions.

1. $2x^2+5x-7=0$.

2. $3x^2-7x+2=0$.

3. $5x^2-2x=16$.

4. $3x=200-x^2$.

5. $6x^2+7x=42$.

6. $x^2+\frac{1}{2}x+\frac{1}{7}=0$.

7. $2x^2+\sqrt{3}\,x+\sqrt{5}=0$.

8. $x^2+px=q$.

9. Show that in the quadratic $x^2+mx+n=0$ the sum of the roots is $-m$ and the product of the roots is n. This general result is important and may be stated in words as follows:

If in a quadratic the coefficient of x^2 is 1, the sum of the solutions will be the coefficient of x with its sign changed, while the product of the solutions will be the remaining term. Explain and illustrate by means of the equation $x^2-10x+12=0$.

10. Apply the result stated in Ex. 9 to determine the sum and the product of the solutions of the following equations.

(*a*) $x^2-5x+7=0$.

(*b*) $x^2-4x=10$.

(*c*) $x^2-\frac{1}{2}x=2$.

(*d*) $x^2-\sqrt{2}\,x+\sqrt{3}=0$.

(*e*) $x^2-(a+b)x+ab=0$.

(*f*) $2x^2+3x-4=0$.

[Hint. First divide through by 2.]

70. Formation of Quadratics Hav'ng Given Solutions.

Example 1. Form the quadratic whose solutions are 1 and $-\frac{1}{2}$.

Solution. If $x=1$, then $x-1=0$; if $x=-\frac{1}{2}$, then $x+\frac{1}{2}=0$.

Hence the equation $(x-1)(x+\frac{1}{2})=0$, or $x^2-\frac{1}{2}x-\frac{1}{2}=0$, will be satisfied when either $x=1$ or $x=-\frac{1}{2}$ (§ 52).

The desired quadratic is therefore

$$x^2-\tfrac{1}{2}x-\tfrac{1}{2}=0, \text{ or } 2x^2-x-1=0. \quad \textit{Ans.}$$

Similarly, if the given solutions are any numbers, as a and b, the equation having these as solutions is obtained by subtracting a from x and b from x, then multiplying the two factors thus obtained together and placing the result equal to 0; that is, the desired equation is $(x-a)(x-b)=0$.

EXERCISES

Form the quadratics whose roots are

1. $2, 3$.

2. $-2, -3$.

3. $4, \frac{1}{4}$.

4. $12, -5$.

5. $\frac{1}{2}, -\frac{1}{4}$.

6. $\sqrt{2}, \sqrt{3}$.

[HINT. Write answer in the form $x^2-(\sqrt{2}+\sqrt{3})x+\sqrt{6}=0$.]

7. $\sqrt{8}, -\sqrt{2}$.

8. $\frac{1}{2}\sqrt{5}, -\frac{1}{2}$.

9. $3m, -5m$.

10. $2a-b, 2a+b$.

11. $3+\sqrt{2}, 3-\sqrt{2}$.

12. $2-\sqrt{5}, 2+\sqrt{5}$.

13. $2\pm\sqrt{3}$.

14. $-\frac{1}{2}(3\pm\sqrt{6})$.

15. $\frac{1}{2}(-1\pm\sqrt{2})$.

16. $m(2\pm 2\sqrt{5})$.

17. Show that if in any quadratic, as $ax^2+bx+c=0$, one root is double the other, then the relation $2b^2=9ac$ must exist among the coefficients, a, b, and c.

[HINT. Let r be one root. Then, by what the problem assumes, the other root is $2r$. Now form the quadratic having r and $2r$ as roots, and examine its coefficients.]

18. Show that if in any quadratic, as $ax^2+bx+c=0$, one root is three times the other, then we must have $4ac=b^2-9a^2$.

19. Find the relation which must hold between a, b, and c in order that the roots of the quadratic $ax^2+bx+c=0$ may be to each other in the ratio $2:3$.

[HINT. Use the Rule of § 69.]

CHAPTER XIII

IMAGINARY NUMBERS

71. Preliminary Statement. Just as $\sqrt{2}$ means the number whose square is 2; that is, $(\sqrt{2})^2=2$, so $\sqrt{-2}$ means the number whose square is -2; that is, $(\sqrt{-2})^2=-2$. The latter case differs essentially, however, from the former, because we cannot conceive of any number, positive or negative, whose square gives a *negative* result, like -2. In fact, this would seem to contradict the law of signs [§ 2 (*d*)], according to which the square of either a positive or negative quantity is always positive. The explanation is that $\sqrt{-2}$ belongs to an altogether new class of numbers, called ***imaginary numbers***, so that we cannot expect to think of them in the way just mentioned; namely, as though they were *real* numbers. Imaginary numbers first came to our notice in § 58, where we found that the very simple quadratic $x^2=-1$ has the two imaginary roots $x=\sqrt{-1}$ and $x=-\sqrt{-1}$.

Imaginary numbers have certain definite properties. They may be added, subtracted, multiplied, divided, etc., in ways which will be explained in the present chapter.

72. The Imaginary Unit. Every pure imaginary number (§ 66) may be expressed as the product of a certain real number multiplied by $\sqrt{-1}$. For this reason, $\sqrt{-1}$ is called

the ***imaginary unit***, and for convenience is represented by the letter i.

Thus $\sqrt{-4}=\sqrt{4(-1)}=2\sqrt{-1}=2\,i.$

$$\sqrt{-27}=\sqrt{27(-1)}=\sqrt{27}\,\sqrt{-1}=3\sqrt{3}\,i.$$

$$\sqrt{-8}=\sqrt{8(-1)}=\sqrt{8}\,\sqrt{-1}=2\sqrt{2}\,i.$$

$$\sqrt{-x^2}=\sqrt{x^2(-1)}=x\sqrt{-1}=xi.$$

A pure imaginary number when thus written as a real number multiplied by i is said to be *expressed in terms of* i.

EXERCISES

Write each of the following expressions in terms of i.

1. $\sqrt{-16}.$
2. $\sqrt{-25}.$
3. $\sqrt{-18}.$
4. $\sqrt{-24}.$
5. $\sqrt{-4\,x}.$
6. $\sqrt{-100\,m^2}.$
7. $\sqrt{-49\,a^2b^2}.$
8. $\sqrt{-50\,x^2}.$
9. $\sqrt{-\frac{1}{4}}.$

Solution of Ex. 9. $\sqrt{\frac{1}{4}(-1)}=\sqrt{\frac{1}{4}}\sqrt{-1}=\frac{1}{2}\,i.$ *Ans.*

10. $\sqrt{-\frac{1}{9}}.$
11. $\sqrt{-\frac{4}{9}}.$
12. $\sqrt{-\frac{25}{16}}.$
13. $\sqrt{-\frac{81}{100}\,x^4}.$
14. $\sqrt{-\frac{27}{4}}.$

Solution of Ex. 14. $\sqrt{-\dfrac{27}{4}}=\sqrt{\dfrac{27}{4}}\,\sqrt{-1}=\dfrac{3\sqrt{3}}{2}\,i.$ *Ans.*

15. $\sqrt{-\frac{12}{25}}.$
16. $\sqrt{-\frac{45}{81}}.$

73. Addition and Subtraction of Pure Imaginary Numbers.

EXAMPLE. Add $\sqrt{-9}$ and $\sqrt{-25}$ and express the result in terms of i.

SOLUTION.

$$\sqrt{-9}+\sqrt{-25}=\sqrt{9}\sqrt{-1}+\sqrt{25}\sqrt{-1}=3\,i+5\,i=8\,i. \quad \textit{Ans.}$$

EXERCISES

Simplify each of the following expressions, obtaining the result in terms of i.

1. $\sqrt{-4}+\sqrt{-36}$.
2. $\sqrt{-49}+\sqrt{-16}$.
3. $\sqrt{-81}+\sqrt{-64}+\sqrt{-100}$.
4. $\sqrt{-8}+\sqrt{-27}$. $(2\sqrt{2}+3\sqrt{3})i$. *Ans.*
5. $\sqrt{-20}-\sqrt{-18}$.
6. $\sqrt{-1}+\sqrt{-12}-\sqrt{-45}$.
7. $\sqrt{-x^2}+\sqrt{-4\,x^2}+\sqrt{-9\,x^2}$.
8. $\sqrt{-16\,a^2}+\sqrt{-100\,a^2}-\sqrt{-81\,a^2}$.
9. $\sqrt{-25\,x^2y^2}+\sqrt{-225\,x^2y^2}+\sqrt{-625\,x^2y^2}$.
10. $\sqrt{-32\,m^2}+\sqrt{-20\,m^2}-\sqrt{-27\,m^2}$.
11. $\sqrt{-24\,h^2k^2}+2\sqrt{-6\,h^2k^2}+3\sqrt{-54\,h^2k^2}$.
12. $\sqrt{-8}+a\sqrt{-2}-\sqrt{-98}-3\sqrt{-3\,a^2}$.

74. Simplification of Complex Numbers.

EXAMPLE. Simplify $2+\sqrt{-9}$.

SOLUTION. $2+\sqrt{-9}=2+3\,i$. *Ans.*

In this exercise we have a real number, 2, to which is added the pure imaginary number, $\sqrt{-9}$, thus giving a complex number (§ 59).

EXERCISES

Simplify each of the following complex numbers.

1. $5+\sqrt{-81}$.
2. $6-2\sqrt{-27}$.
3. $\frac{1}{3}+\sqrt{-\frac{8}{9}}$.
4. $\frac{1}{3}-\sqrt{-\frac{10}{9}}$.
5. $\frac{1}{4}-\sqrt{-\frac{9}{16}}$.
6. $-\frac{2}{3}+\sqrt{-\frac{15}{16}}$.

7. Show that the quadratic equation $x^2-4x+13=0$ has as its solutions $x=2\pm 3i$.

[HINT. Solve as in § 56.]

8. Find the solutions of the equation $4(2x-5)=x^2$.

75. Multiplication of Imaginary Numbers.

EXAMPLE 1. Multiply $\sqrt{-3}$ by $\sqrt{-4}$.

SOLUTION. $\sqrt{-3}\cdot\sqrt{-4}=\sqrt{3}\,i\cdot\sqrt{4}\,i=\sqrt{12}\cdot i^2=\sqrt{12}(\sqrt{-1})^2$
$= -\sqrt{12}=-2\sqrt{3}.$ *Ans.*

Note that the process of multiplication consists in first expressing each number in terms of the unit i, then making use of the fact that (since $i=\sqrt{-1}$) we may write -1 for i^2.

EXAMPLE 2. Multiply $2+\sqrt{-3}$ by $4-\sqrt{-3}$.

SOLUTION. $(2+\sqrt{-3})(4-\sqrt{-3})=(2+\sqrt{3}\,i)(4-\sqrt{3}\,i)$.

$$\begin{array}{l} 2+\sqrt{3}\,i \\ \underline{4-\sqrt{3}\,i} \\ 8+4\sqrt{3}\,i \\ \underline{\quad -2\sqrt{3}\,i-(\sqrt{3})^2 i^2} \\ 8+2\sqrt{3}\,i-3(-1)=11+2\sqrt{3}\,i. \quad \textit{Ans.} \end{array}$$

EXERCISES

Find each of the following indicated products.

1. $\sqrt{-4}\cdot\sqrt{-25}$.
2. $\sqrt{-2}\cdot\sqrt{-18}$.
3. $\sqrt{-10}\cdot\sqrt{-5}$.
4. $\sqrt{-3}\cdot\sqrt{-15}$.
5. $2\sqrt{-2}\cdot 3\sqrt{-2}$.
6. $5\sqrt{-3}\cdot 3\sqrt{-2}$.
7. $\sqrt{-9a^2}\cdot\sqrt{-25a^2}$.
8. $(1+\sqrt{-2})(1-\sqrt{-2})$.
9. $(2+\sqrt{-1})(2-\sqrt{-1})$.
10. $(5-\sqrt{-6})(5+\sqrt{-6})$.
11. $(2+\sqrt{-3})(3+\sqrt{-3})$.
12. $(2+\sqrt{-3})(3+\sqrt{-2})$.
13. $(1-\sqrt{-2})^2$.
14. $(x+\sqrt{-y})(x-\sqrt{-y})$.
15. $a\sqrt{-b}\cdot c\sqrt{-b}$.
16. $a\sqrt{-b}\cdot c\sqrt{-d}$.

76. Division of Imaginary Numbers.

EXAMPLE 1. Find the value of $\sqrt{-27} \div \sqrt{-3}$.

SOLUTION. $\dfrac{\sqrt{-27}}{\sqrt{-3}} = \dfrac{\sqrt{27}\,i}{\sqrt{3}\,i} = \dfrac{\sqrt{27}}{\sqrt{3}} = \sqrt{\dfrac{27}{3}} = \sqrt{9} = 3.$ *Ans.*

EXAMPLE 2. Find the value of $6 \div \sqrt{-3}$.

SOLUTION. $\dfrac{6}{\sqrt{-3}} = \dfrac{6}{\sqrt{3}\,i} = \dfrac{6\,i}{\sqrt{3}(i)^2} = \dfrac{6\,i}{-\sqrt{3}} = \dfrac{6\sqrt{3}\,i}{-3} = -2\sqrt{3}\,i.$ *Ans.*

EXAMPLE 3. Find the value of $3 \div (1 + \sqrt{-2})$.

SOLUTION. $\dfrac{3}{1+\sqrt{-2}} = \dfrac{3}{1+\sqrt{2}\,i} = \dfrac{3(1-\sqrt{2}\,i)}{(1+\sqrt{2}\,i)(1-\sqrt{2}\,i)}$

$= \dfrac{3(1-\sqrt{2}\,i)}{1-2\,i^2} = \dfrac{3(1-\sqrt{2}\,i)}{1+2} = 1 - \sqrt{2}\,i.$ *Ans.*

Note that in Example 2 we rationalized the denominator of $\dfrac{6\,i}{-\sqrt{3}}$ by multiplying both numerator and denominator by the value $\sqrt{3}$. Likewise, in Example 3 we rationalized the denominator of $\dfrac{3}{1+\sqrt{2}\,i}$ by multiplying both numerator and denominator by $1-\sqrt{2}\,i$. In general, any fraction having a denominator of the form $a+\sqrt{-b}$ may be rationalized by multiplying both numerator and denominator by $a-\sqrt{-b}$, which is called the *conjugate imaginary* of $a+\sqrt{-b}$.

EXERCISES

Find the following indicated quotients, rationalizing the denominator in each.

1. $\sqrt{-16} \div \sqrt{-4}$.
2. $\sqrt{-20} \div \sqrt{-5}$.
3. $\sqrt{-45} \div \sqrt{-9}$.
4. $2 \div \sqrt{-6}$.
5. $\sqrt{3} \div \sqrt{-2}$.
6. $\sqrt{-3} \div \sqrt{2}$.

7. $\sqrt{63} \div \sqrt{-7}$.

8. $2\sqrt{-75} \div \sqrt{-3}$.

9. $6\sqrt{-24} \div 2\sqrt{-6}$.

10. $\sqrt{-45\,x^2} \div \sqrt{-5\,x^2}$.

11. $\dfrac{2}{1+\sqrt{-3}}$.

12. $\dfrac{2}{1-\sqrt{-3}}$.

13. $\dfrac{3+2\sqrt{-5}}{3-2\sqrt{-5}}$.

14. $\dfrac{4-2\sqrt{-3}}{1+2\sqrt{-3}}$.

77. Geometric Representation of Complex Numbers.** All real numbers (positive or negative) can be represented as points on a line, as explained in § 1. Similarly, all complex numbers may be represented as points in a *plane*, and it is convenient for many purposes to regard them in this way. Thus, the complex number $5+4\,i$ may be looked upon as lying at the point $(5, 4)$, that is at the point whose abscissa is 5 and whose ordinate is 4. (See § 28.) Likewise, the complex number $-2+3\,i$ lies at $(-2, 3)$; the number $3-2\,i$ lies at $(3, -2)$; and, in general, the number $x+yi$, where x and y are any (real) numbers, lies at the point (x, y). Whenever a plane is used in this way to represent complex numbers, it is called a ***complex plane. The x-axis is called the ***axis of reals*** and the y-axis is called the ***axis of pure imaginaries*** (because the pure imaginary part of each complex number is to be measured parallel to it). The straight line joining any point $x+iy$ to the origin is called the ***radius vector*** of that point, or number.

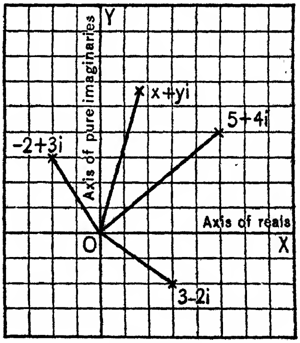


FIG. 39.

Observe that from this point of view all real numbers become represented by the points on the axis of reals, while all pure imaginary numbers become represented by the points on the axis of pure imaginaries, the other points of the plane being then taken up by what are properly the complex numbers.

CHAPTER XIV

SIMULTANEOUS QUADRATIC EQUATIONS

I. One Equation Linear and the Other Quadratic

78. Graphical Solution. In Chapter VI we have seen how we may determine graphically the solution of two linear equations each of which contains the two unknown letters x, y. The method consists in first drawing the graph of each equation, then observing the x and the y of the point where the two graphs intersect (cut each other). The particular pair of values (x, y) thus obtained constitutes the solution.

We often meet with simultaneous equations which are *not* both linear, as for example the two equations

(1) $$x-y=1,$$

(2) $$x^2+y^2=25.$$

Here the first equation is linear (§ 26) but the second is not. In order to solve them, that is in order to find that pair (or pairs) of values of x and y which satisfy both equations, we may proceed graphically as follows.

The graph of (1) is found (as in § 29) to be the straight line shown in Fig. 40.

To draw the graph of (2), we first solve this equation for y. Thus $y^2=25-x^2$. Therefore

(3) $$y=\pm\sqrt{25-x^2}.$$

We now work out a table of pairs values of x and y that will satisfy (3), that is we give x various values in (3) and solve for the corresponding y value. (Compare § 60.) The result is shown below. Observe that to the value $x=0$ there cor-

When $x=$	0	+1	+2	+3	+4	+5
then $y=$	$\pm\sqrt{25}$	$\pm\sqrt{24}$	$\pm\sqrt{21}$	$\pm\sqrt{16}$	$\pm\sqrt{9}$	$\pm\sqrt{0}$
$=$	± 5	± 4.8	± 4.5	± 4	± 3	0

respond the *two* values $y=\pm 5$; similarly to $x=1$ correspond the *two* values $y=\pm 4.8$ (approximately), etc.

Moreover, if we assign to x the *negative* value, $x=-1$, we find in the same way that corresponding to it y has the two values $y=\pm 4.8$. Likewise, for $x=-2$ we find $y=\pm 4.5$, etc., the values of y for any negative value of x being the same each time as for the corresponding positive value of x.

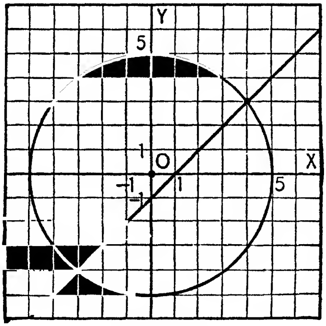


FIG. 40.

Plotting all the points (x, y) thus obtained and drawing the smooth curve through them, the graph is as shown in Fig. 40. This curve is a *circle*, as appears more and more clearly as we plot more and more points (x, y) belonging to the equation (3).

NOTE. The form of (3) shows that there can be no points in the graph having x values greater than 5, for as soon as x exceeds 5 the expression $25-x^2$ becomes *negative* and hence $\sqrt{25-x^2}$ becomes *imaginary*, and there is no point that we can plot corresponding to such a result. Similarly, it appears from (3) that x cannot take values less than -5.

Thus the graph can contain no points lying outside the circle already drawn.

Returning now to the problem of solving (1) and (2), we know (§ 31) that wherever the one graph cuts the other we shall have a point whose x and y form a solution of (1) and (2), that is we shall have a pair of values (x, y) that will satisfy *both* equations at once. From the figure it appears that there are in the present case *two* such points, namely $(x=4, y=3)$ and $(x=-3, y=-4)$. Equations (1) and (2) therefore have the *two* solutions $(x=4, y=3)$ and $(x=-3, y=-4)$. *Ans.*

CHECK. For the solution $(x=4, y=3)$ we have $x-y=4-3=1$, and $x^2+y^2=16+9=25$, as required.

For the solution $(x=-3, y=-4)$ we have $x-y=-3-(-4)=1$, and $x^2+y^2=9+16=25$, as required.

The following are other examples of the graphical study of non-linear simultaneous equations.

EXAMPLE 1. Solve the system

(4) $$2x-9y+10=0,$$

(5) $$4x^2+9y^2=100.$$

SOLUTION. The straight line representing the graph of (4) is drawn readily.

To obtain the graph of (5), we have $9y^2=100-4x^2$. Hence

$$y^2=\frac{1}{9}(100-4x^2)=\frac{4}{9}(25-x^2),$$

and therefore

(6) $$y=\pm\frac{2}{3}\sqrt{25-x^2}.$$

Corresponding to (6), we find the following table:

When $x=$	0	+1	+2	+3	+4	+5	greater than +5
then $y=$	$\pm\frac{2}{3}\sqrt{25}$	$\pm\frac{2}{3}\sqrt{24}$	$\pm\frac{2}{3}\sqrt{21}$	$\pm\frac{2}{3}\sqrt{16}$	$\pm\frac{2}{3}\sqrt{9}$	$\pm\frac{2}{3}\sqrt{0}$	imaginary
$=$	$\pm\frac{2}{3}(5)$	$\pm\frac{2}{3}(4.8)$	$\pm\frac{2}{3}(4.5)$	$\pm\frac{2}{3}(4)$	$\pm\frac{2}{3}(3)$	± 0	imaginary
$=$	± 3.3	± 3.2	± 3.0	± 2.6	± 2	0	imaginary

For any negative value of x, the y-values are the same as for the corresponding positive value of x. (See the solution of (1) and (2).)

The graph thus obtained for (6), or (5), is an oval shaped curve. It belongs to a general class of curves called *ellipses*.

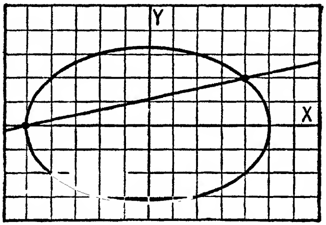


Fig. 41.

The two graphs are seen to intersect at the points

$$(x=4, y=2) \text{ and } (x=-5,\ y=0).$$

Therefore the desired solutions of (4) and (5) are $(x=4,\ y=2)$ and $(x=-5,\ y=0)$. *Ans.*

Example 2. Solve the system

$$(7) \qquad 2x-y=-2,$$

$$(8) \qquad xy=4.$$

Solution. The graph of (7) is the straight line shown in Fig. 42.

To obtain the graph of (8), we have

$$(9) \qquad y=\frac{4}{x},$$

from which we obtain the following table:

When $x =$	8	7	6	5	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
then $y =$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{2}{3}$	$\frac{4}{5}$	1	$\frac{4}{3}$	2	4	8	12	16	20

This table concerns only positive values of x, but it appears from (9) that for any negative value of x the appropriate y value is the negative of that for the corresponding positive value of x.

The graph thus obtained for (9), or (8), consists of two open curves, each indefinitely long, situated as in Fig. 42. These taken together (that is, regarded as one curve) form what is known as a *hyperbola* (pronounced hy-per′bo-la). The part (branch) lying to the right of the y-axis corresponds to the above table, while the other branch corresponds to the negative x values.

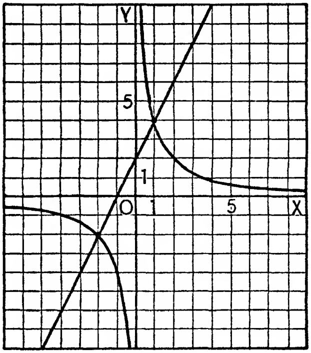


Fig. 42.

The two graphs are seen to intersect in the points $(x=1,\ y=4)$ and $(x=-2,\ y=-2)$.

Therefore the desired solutions of (7) and (8) are $(x=1,\ y=4)$ and $(x=-2,\ y=-2)$. *Ans.*

Note. Ellipses and hyperbolas are extensively considered in the branch of mathematics called *analytic geometry* — a study which may be pursued after a course in plane geometry and a course in algebra equivalent to that in this book. It is usually taught during the first year of college mathematics.

EXAMPLE 3. Consider graphically the system

$$x+y=10, \tag{10}$$

$$x^2+y^2=25. \tag{11}$$

SOLUTION. The graph of (10) is found in the usual manner, and is represented by the straight line in Fig. 43. The graph of (11) has already been worked out (see discussion of (2)), being a circle of radius 5 with center at the origin. The peculiarity to be especially observed here is that these two graphs *do not intersect.* This means (as it naturally must) that there are no *real* solutions to the system (10) and (11); in other words, the only possible solutions are *imaginary.*

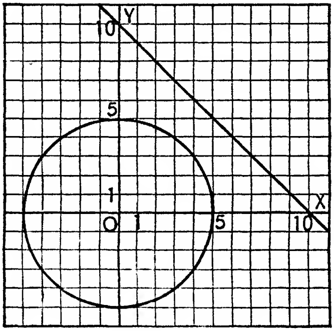


FIG. 43.

Likewise, whenever *any* two graphs fail to intersect, we may be assured at once that the only solutions their equations can have are imaginary. The system (10) and (11) and other such systems will be considered further in the next article.

EXERCISES

Draw the graphs for the following systems and use your result to determine the solutions whenever they are real.

1. $\begin{cases} x+y=6, \\ x^2+y^2=20. \end{cases}$

2. $\begin{cases} x+3\,y=-5, \\ 4\,x^2+9\,y^2=100. \end{cases}$

[HINT. See Example 1, § 78.]

3. $\begin{cases} x-2y=-1, \\ x^2+4y^2=25. \end{cases}$

4. $\begin{cases} 2x+y=4, \\ xy=-16. \end{cases}$

[HINT. See Example 2, § 78.]

5. $\begin{cases} x^2-y^2=16, \\ 5x-3y=0. \end{cases}$

6. $\begin{cases} x^2-y^2=16, \\ y=2x. \end{cases}$

7. $\begin{cases} x+y=2, \\ y=x^2. \end{cases}$

8. $\begin{cases} 2x+y=1, \\ y=4x^2+2x+1. \end{cases}$

79. Solution by Elimination. Let us consider again the system (1) and (2) of § 78.

(1) $$x-y=1,$$
(2) $$x^2+y^2=25.$$

Instead of solving this system graphically, we may solve it by elimination, that is by the process employed with two linear equations in § 33.

Thus we have from (1)

(3) $$y=x-1.$$

Substituting this value of y in (2), thus eliminating y from (2), we obtain

$$x^2+(x-1)^2=25, \text{ or } x^2+x^2-2x+1=25,$$

or
$$2x^2-2x-24=0,$$

or, dividing through by 2,

(4) $$x^2-x-12=0.$$

Solving (4) by formula (§ 56), gives as the two roots

$$x=\frac{-(-1)+\sqrt{(-1)^2-4(1)(-12)}}{2}=\frac{1+\sqrt{1+48}}{2}=\frac{1+7}{2}=4,$$

and

$$x=\frac{-(-1)-\sqrt{(-1)^2-4(1)(-12)}}{2}=\frac{1-\sqrt{1+48}}{2}=\frac{1-7}{2}=-3.$$

When x has the first of these values, namely 4, we see from (3) that y must have the value $y=4-1$, or 3.

Similarly, when x takes on its other value, namely -3, we see that y has the value $y=-3-1$, or -4.

The solutions of the system (1) and (2) are, therefore, $(x=4,\ y=3)$ and $(x=-3,\ y=-4)$. *Ans.*

Observe that these results agree with those obtained graphically for (1) and (2) in § 78.

Further applications of this method are made in the examples that follow.

EXAMPLE 1. Solve the system

$$2x+y=4, \tag{5}$$

$$x^2+y^2=12. \tag{6}$$

SOLUTION. From (5),

$$y=4-2x. \tag{7}$$

Substituting this expression for y in (6), we find

$$x^2+(16-16x+4x^2)=12,$$

or

$$5x^2-16x+4=0. \tag{8}$$

The two roots of (8), as determined by formula (§ 56), are

$$x=\frac{-(-16)\pm\sqrt{(-16)^2-4(5)(4)}}{2(5)}=\frac{16\pm\sqrt{256-80}}{10}=\frac{16\pm\sqrt{176}}{10}$$

$$=\frac{16\pm4\sqrt{11}}{10}=\frac{8\pm2\sqrt{11}}{5}.$$

The first of these values, namely $x=(8+2\sqrt{11})/5$, when substituted in (7), gives as its corresponding value of y,

$$y=4-\frac{16+4\sqrt{11}}{5}=\frac{4-4\sqrt{11}}{5}.$$

The second value, namely $x=(8-2\sqrt{11})/5$, when substituted in (7), gives as its corresponding value of y,

$$y=4-\frac{16-4\sqrt{11}}{5}=\frac{4+4\sqrt{11}}{5}.$$

Hence the desired solutions are

$$\begin{cases} x=\dfrac{8+2\sqrt{11}}{5}, \\ y=\dfrac{4-4\sqrt{11}}{5}, \end{cases} \quad \text{and} \quad \begin{cases} x=\dfrac{8-2\sqrt{11}}{5}, \\ y=\dfrac{4+4\sqrt{11}}{5}. \end{cases}$$

To obtain the approximate values of the numbers thus obtained we have $\sqrt{11} = 3.31662$ (tables), and hence the above solutions reduce to the forms

$$\begin{cases} x = 2.9266, \\ y = -1.8533, \end{cases} \quad \text{and} \quad \begin{cases} x = 0.2734, \\ y = 3.4533. \end{cases}$$

These are the solutions of the system (5), (6), *correct to four places of decimals*, which is sufficient for ordinary work.

NOTE. It may be remarked that, while the graphical method of solution described in § 78 is very instructive in showing how many solutions a given system will have, and what their geometric significance is, it does not usually afford a ready means of determining the *exact values* of the solutions. This is illustrated in the example just solved, where, if the graphs of (5) and (6) be drawn, they will intersect at points whose x and y contain the surd $\sqrt{11}$ (as the above solution shows), and it would be difficult to measure off any such values accurately on the scale of the diagram. In fact, it would be practically impossible to determine graphically the solutions of (5) and (6) correct to three or even two places of decimals, yet this degree of approximation was easily obtained above by the method of elimination. For such reasons, it is preferable, whenever one is concerned only with finding the *values* of solutions, to proceed from the beginning by the method of elimination.

EXAMPLE 2. Solve the system

(9) $$x + y = 10,$$

(10) $$x^2 + y^2 = 25.$$

SOLUTION. From (9),

(11) $$y = 10 - x.$$

Substituting this expression in (10),

$$x^2 + (100 - 20x + x^2) = 25,$$

or

(12) $$2x^2 - 20x + 75 = 0.$$

Solving (12) by formula, we find its solutions to be

$$x = \frac{20 + \sqrt{-200}}{10} \quad \text{and} \quad \frac{\quad}{10}$$

Since these x-values contain the square root of the negative quantity -200, they are imaginary (§ 58). The y-values are also

imaginary, as appears by substituting the x-values just found into (9), which gives the results

$$y=\frac{80-\sqrt{-200}}{10} \quad \text{and} \quad y=\frac{80+\sqrt{-200}}{10}.$$

The desired solutions of the systems (9), (10) are therefore

$$\begin{cases} x=\dfrac{20+\sqrt{-200}}{10}, \\ y=\dfrac{80-\sqrt{-200}}{10}, \end{cases} \quad \text{and} \quad \begin{cases} x=\dfrac{20-\sqrt{-200}}{10}, \\ y=\dfrac{80+\sqrt{-200}}{10}. \end{cases}$$

This result should now be contrasted with what we saw in Example 3 of § 78 regarding this same system (9) and (10). There we found *graphically* that the solutions must be imaginary because the graphs failed to intersect, but we could not find the actual imaginary numbers which form the solutions. This we have now been able to do, however, by the method of elimination. The method of elimination enables one to determine imaginary as well as real solutions in all similar cases.

EXERCISES

Solve each of the following systems by the method of elimination, and, in case surds are present, find each solution correct to two places of decimals by use of the tables.

1. $\begin{cases} x+y=-1, \\ x^2+y^2=13. \end{cases}$

2. $\begin{cases} x-2y=-1, \\ x^2+4y^2=25. \end{cases}$

3. $\begin{cases} x-2y=2, \\ x^2+4y^2=25. \end{cases}$

4. $\begin{cases} 2x+y=4, \\ xy=-6. \end{cases}$

5. $\begin{cases} 2x+y=2, \\ xy=-6. \end{cases}$

6. $\begin{cases} x^2-2y^2=8, \\ x-2y=3. \end{cases}$

7. $\begin{cases} x^2-2y^2=-8, \\ x-2y=-3. \end{cases}$

8. $\begin{cases} 3x^2-xy-5y^2=5, \\ 3x-5y=1. \end{cases}$

9. $\begin{cases} \dfrac{4x}{3y}+\dfrac{2y}{5x}=\dfrac{34}{15}, \\ 2x-5y=-4. \end{cases}$

10. $\begin{cases} \dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}=\dfrac{5}{6}, \\ 2x+5y=5. \end{cases}$

II. Neither Equation Linear

***80. Two Quadratic Equations.** In each of the systems considered in §§ 78, 79 one of the two given equations was linear. However, the same methods of solving may often be employed in case *neither* equation is linear. In such cases *four* solutions may be present instead of two.

Example 1. Solve the system

(1) $$9x^2+16y^2=160,$$

(2) $$x^2-y^2=15.$$

Solution. Here only x^2 and y^2 appear and we begin by finding their values. Thus, multiplying (2) through by 16 and adding the result to (1), we eliminate y^2 and find that $25x^2=400$, or

(3) $$x^2=16.$$

Substituting this value of x in (2), we find

(4) $$y^2=1.$$

From (3) and (4) we now obtain

(5) $$x=\pm 4 \quad \text{and} \quad y=\pm 1.$$

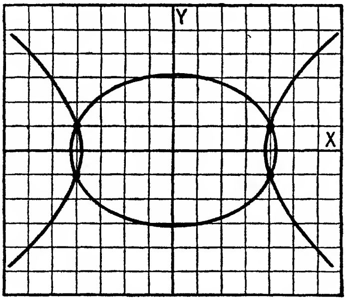


Fig. 44.

Forming all the pairs of values x, y that can come from (5), we obtain as our desired solutions

$(x=4,\ y=1)$; $(x=-4,\ y=1)$;
$(x=4,\ y=-1)$; and
$(x=-4,\ y=-1)$. *Ans.*

Check. Each of these pairs of values of x and y is immediately seen to satisfy both (1) and (2). Let the pupil thus check each pair.

When considered graphically, equation (1) gives rise to an ellipse (compare § 78, Ex. 1), while (2) gives a hyperbola situated as shown in the diagram. These two curves intersect in *four* points which correspond to the four solutions just obtained.

EXAMPLE 2. Solve the system

(7) $$x^2+y^2=25,$$

(8) $$xy=-12.$$

SOLUTION. Here we cannot proceed as in Example 1 because we cannot find readily the values of x^2 and y^2. But if we multiply (8) by 2 and add the result to (7), we obtain

(9) $$x^2+2xy+y^2=1.$$

Taking the square root of both members of (9) gives

(10) $$x+y=\pm 1.$$

Similarly, multiplying (8) by 2 and *subtracting* the result from (7),

$$x^2-2xy+y^2=49,$$

and hence

(11) $$x-y=\pm 7.$$

Taking account of the two choices of sign in (10) and (11), we see that they give rise to the four simple (linear) systems

(*a*) $x+y=1,\ x-y=7;$
(*b*) $x+y=-1,\ x-y=7;$
(*c*) $x+y=1,\ x-y=-7;$
(*d*) $x+y=-1,\ x-y=-7;$

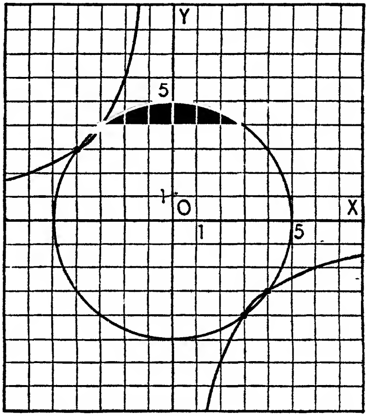


FIG. 45.

Thus we have replaced the original system (7) and (8) by the four simple systems (*a*), (*b*), (*c*), and (*d*), each of which may be immediately solved by elimination, as in §§ 33, 34. Since the solutions of (*a*), (*b*), (*c*), (*d*) are respectively ($x=4$, $y=-3$), ($x=3$, $y=-4$), ($x=-3$, $y=4$), and ($x=-4$, $y=3$), we conclude that these are the desired solutions of (7) and (8). *Ans.*

The graphical significance of these solutions is shown in Fig. 45, where the circle $x^2+y^2=25$ is cut by the hyperbola $xy=-12$ in *four* points that correspond to the four solutions just found.

CHECK. That these four solutions each satisfy (7) and (8) appears at once by trial.

While no general rule can be stated for solving two equations neither of which is linear, the following observation may be made. Unless the equations can be solved readily for x^2 and y^2 (as in Example 1), the system should first be carefully examined with a view to making such combinations of the given equations as will yield one or more new systems each of which can be solved (as in Example 2) by methods already familiar. All solutions obtained in this way should be checked, since false combinations of the x- and y-values are frequently made by beginners when the work becomes at all complicated.

EXERCISES

Solve each of the following systems, and draw a diagram for each of the first three to show the geometric meaning of your solutions.

1. $\begin{cases} x^2+y^2=25, \\ x^2-y^2=7. \end{cases}$

2. $\begin{cases} 4x^2+9y^2=100, \\ 2x^2-y^2=28. \end{cases}$

3. $\begin{cases} x^2+y^2=20, \\ xy=8. \end{cases}$

4. $\begin{cases} x^2+xy=36, \\ y^2+xy=45. \end{cases}$

[Hint for Ex. 4. First add, then subtract the two equations, thus showing that the given system is equivalent to two others, namely

$$\begin{cases} x+y=9, \\ x^2-y^2=-9, \end{cases} \quad \text{and} \quad \begin{cases} x+y=-9, \\ x^2-y^2=-9. \end{cases}$$

Now solve each of these systems as in § 80.]

5. $\begin{cases} 4x^2+9y^2=85, \\ xy=3. \end{cases}$

6. $\begin{cases} x^2+xy+y^2=79, \\ x^2-xy+y^2=37. \end{cases}$

7. $\begin{cases} xy-6=0, \\ x^2+y^2=xy+7. \end{cases}$

8. $\begin{cases} u^2+uv=45, \\ u^2-uv=5. \end{cases}$

9. $\begin{cases} s^2-t^2=15, \\ s=4t^2. \end{cases}$

10. $\begin{cases} x-xy=9, \\ y+2xy=-20. \end{cases}$

CARDAN

(*Girolamo Cardan,* 1501–1576)

An equation of the *third* degree (cubic equation) has *three* roots and these can be found only by methods which are more powerful than those employed in the study of quadratics. *Cardan* was the first to obtain and publish a method for solving such equations. His methods are also sufficient to solve any pair of simultaneous quadratics, but are too advanced to be given in this book.

***81. Systems Having Special Forms.** The systems of equations considered in §§ 79, 80 illustrate the usual and more simple types such as one commonly meets in practice. It is possible, however, to solve more complicated systems provided they are of certain prescribed forms. We shall here consider only two such type forms.

I. *When one (or both) of the given equations is of the form*

$$ax^2+bxy+cy^2=0,$$

where the coefficients a, b, c are such that the expression $ax^2+bxy+cy^2$ can be factored into two linear factors.

Example. Solve the system

(1) $$x^2+2x-y=7,$$
(2) $$x^2-xy-2y^2=0.$$

Solution. Here we see that (2) is of the form mentioned above, since $x^2-xy-2y^2$ can be factored (as in §12 (e)) into $(x-2y)(x+y)$. (2) may thus be written in the form

(3) $$(x-2y)(x+y)=0.$$

It follows (§ 52) that either

$$x-2y=0, \quad \text{or} \quad x+y=0.$$

Hence the system (1), (2) may be replaced by the two following systems:

$$\begin{cases} x^2+2x-y=7, \\ x-2y=0, \end{cases}$$

and

$$\begin{cases} x^2+2x-y=7, \\ x+y=0. \end{cases}$$

Each of these two systems may now be solved as in § 80, and we thus find that the solutions of the first system are

$$(x=2, \ y=1) \text{ and } (x=-\tfrac{7}{2}, y=-\tfrac{7}{4})$$

while the solutions of the second system are

$$\begin{cases} x=\frac{1}{2}(-3+\sqrt{37}), \\ y=\frac{1}{2}(3-\sqrt{37}), \end{cases}$$

and

$$\begin{cases} x=\frac{1}{2}(-3-\sqrt{37}), \\ y=\frac{1}{2}(3+\sqrt{37}). \end{cases}$$

The desired solutions of (1) and (2) consist, therefore, of these four solutions just obtained. *Ans.*

II. *When both the given equations are of the form*

$$ax^2+bxy+cy^2=d,$$

where a, b, c, and d have any given values (0 *included*).

Example. Solve the system

$$x^2-xy+y^2=3, \tag{4}$$

$$x^2+2\,xy=5. \tag{5}$$

Solution. Let v stand for the ratio x/y; that is, let us set

$$\frac{x}{y}=v.$$

Then

$$x=vy. \tag{6}$$

Substituting in (4),

$$v^2y^2-vy^2+y^2=3. \tag{7}$$

Substituting in (5),

$$v^2y^2+2\,vy^2=5. \tag{8}$$

Solving (7) for y^2,

$$y^2=\frac{3}{v^2-v+1}. \tag{9}$$

Solving (8) for y^2,

$$y^2=\frac{5}{v^2+2\,v}. \tag{10}$$

Equating the values of y^2 given by (9) and (10),

$$\frac{5}{v^2+2\,v}=\frac{3}{v^2-v+1}.$$

Clearing of fractions,

$$2\,v^2-11\,v+5=0. \tag{11}$$

Solving (11) by formula (§ 56),

$$v=\frac{11\pm\sqrt{121-40}}{4}=\frac{11\pm\sqrt{81}}{4}=\frac{11\pm 9}{4}.$$

Therefore $v=5$, or $v=\frac{1}{2}$. Substituting 5 for v in (9), or (10),

$$y^2=\tfrac{1}{7}.$$

Hence

$$y=+\frac{1}{\sqrt{7}},\text{ or }-\frac{1}{\sqrt{7}}.$$

Substituting $\frac{1}{2}$ for v in (9) or (10), $y^2=4$. Hence $y=+2$, or -2.

The only values that y can have are, therefore, $1/\sqrt{7}$, $-1/\sqrt{7}$, 2, and -2.

Since $x=vy$ (see (6)), the value of x to go with $y=1/\sqrt{7}$ is $x=5(1/\sqrt{7})=5/\sqrt{7}$. Similarly, when $y=-1/\sqrt{7}$ we have $x=5(-1/\sqrt{7})=-5/\sqrt{7}$. Likewise, when $y=2$ (in which case $v=\frac{1}{2}$, as shown on p. 136) then $x=\frac{1}{2}\cdot 2=1$, and when $y=-2$, then $x=\frac{1}{2}(-2)=-1$.

Therefore the only solutions which the system (4), (5) can have are $(x=5/\sqrt{7},\ y=1/\sqrt{7})$; $(x=-5/\sqrt{7},\ y=-1/\sqrt{7})$; $(x=1,\ y=2)$; $(x=-1,\ y=-2)$; and it is easily seen by checking that each of these is a solution. *Ans.*

82. Conclusion. Every system of equations considered in this chapter has been such that we could solve it by finally solving one or more simple quadratic equations. We have examined only special types, however, and the student should not conclude that all pairs of simultaneous quadratics can be solved so simply.

MISCELLANEOUS EXERCISES

Solve the following simultaneous quadratics. The star (*) indicates that the exercise depends upon § 81.

1. $\begin{cases} x^2+y^2=25, \\ x+y=1. \end{cases}$

2. $\begin{cases} 3x^2-xy-5y^2=5, \\ 3x-5y=1. \end{cases}$

3. $\begin{cases} x+y=2, \\ \dfrac{2}{x}+\dfrac{3}{y}=6. \end{cases}$

4. $\begin{cases} xy+2x=5, \\ 2xy-y=3. \end{cases}$

[HINT TO EX. 4. First eliminate xy between the two equations so as to obtain a linear equation between x and y.]

5. $\begin{cases} x^2+2x-y=5, \\ 2x^2-3x+2y=8. \end{cases}$

6. $\begin{cases} x^4-y^4=369, \\ x^2-y^2=9. \end{cases}$

7. $\begin{cases} 4x^2+y^2=61, \\ 2x^2+3y^2=93. \end{cases}$

8. $\begin{cases} x^2+y^2=100, \\ (x+y)^2=196. \end{cases}$

[HINT TO EX. 6. Divide the first equation by the second.]

9. $\begin{cases} x^2-xy=6, \\ x^2-y^2=8. \end{cases}$

***12.** $\begin{cases} 2x^2+xy-y^2=0, \\ 2x^2+y=1. \end{cases}$

10. $\begin{cases} x^2+y^2+x+3y=18, \\ xy-y=12. \end{cases}$

***13.** $\begin{cases} 2x^2-3y-y^2=8, \\ 6x^2-5xy-6y^2=0. \end{cases}$

11. $\begin{cases} x^2+xy+y^2=19, \\ x^2y^2=144. \end{cases}$

***14.** $\begin{cases} xy+2y^2=8, \\ x^2+2xy=12. \end{cases}$

***15.** $\begin{cases} x^2-xy-y^2=20, \\ x^2-3xy+2y^2=8. \end{cases}$

APPLIED PROBLEMS

In working the following problems, let x and y represent the two unknown quantities, then form two simultaneous equations and solve them. If surds occur, find their approximate values by the tables.

1. The sum of two numbers is 13 and the difference of their squares is 91. Find the numbers.

2. A piece of wire 48 inches long is bent into the form of a right triangle whose hypotenuse is 20 inches long. What are the lengths of the sides? (See Ex. 14 (*d*), p. 6.)

3. If it takes 26 rods of fence to inclose a rectangular garden containing $\frac{1}{4}$ of an acre, what are the length and breadth?

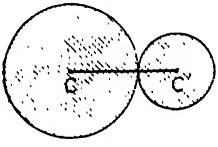


FIG. 46.

4. Figure 46 shows two circles just touching (tangent to) each other, the smaller one being outside the larger one. If their combined area is $15\frac{5}{7}$ square feet and the distance CC' between the two centers is 3 feet, find the radius of each circle. (Take $\pi=\frac{22}{7}$.)

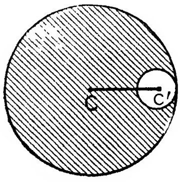


FIG. 47.

5. Work Ex. 4 in case the circles touch on the *inside* of the larger one, taking the shaded area to be 110 square feet and CC' to be 5 feet.

6. Do *positive integers* exist differing by 5 and such that the difference of their squares is 45? If so, find them.

7. Answer the question in Ex. 6 in case the difference of the squares is taken to be 10, other conditions remaining the same.

8. The area of a certain triangle is 160 square feet, and its altitude is twice as long as its base. Find, correct to three decimal places, the base and the altitude. (See Ex. 14 (*a*), p. 6.)

9. The area of a rectangular lot is 2400 square feet, and the diagonal across it measures 100 feet. Find, correct to three decimal places, the length and breadth.

10. The mean proportional between two numbers is 2 and the sum of their squares is 10. What are the numbers? (See Ex. 6, page 80.)

11. The dimensions of a rectangle are 5 feet by 2 feet. Find the amounts (correct to two decimal places) by which each dimension must be changed, and how, in order that both the area and the perimeter shall be doubled.

12. Two men working together can complete a piece of work in 6 days. If it would take one man 5 days longer than the other to do the work alone, in how many days can each do it alone? (Compare Ex. 9, p. 56.)

13. The fore wheel of a carriage makes 28 revolutions more than the rear wheel in going 560 yards, but if the circumference of each wheel be increased by 2 feet, the difference would be only 20 revolutions. What is the circumference of each wheel?

14. A sum of money on interest for one year at a certain rate brought $7.50 interest. If the rate had been 1% less and the principal $25 more, the interest would have been the same. Find the principal and the rate.

15. A man traveled 30 miles. If his rate had been 5 miles more per hour, he could have made the journey in 1 hour less time. Find his time and rate.

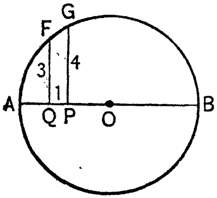


Fig. 48.

16. Figure 48 shows a circle within which a diameter AB has been drawn. At a certain point P on AB the perpendicular PG measures 4 inches, while at the point Q, which is 1 unit from P, the perpendicular QF measures 3 inches. How long is the diameter AB?

[Hint. Let $x = AP$, $y = PB$. Find x and y, using the fact stated in Ex. 6, page 80, then take their sum.]

17. Show that the formulas for the length l and the width w of the rectangle whose perimeter is a and whose area is b are

$$l = \tfrac{1}{4}(a + \sqrt{a^2 - 16\,b}),\quad w = \tfrac{1}{4}(a - \sqrt{a^2 - 16\,b}).$$

18. Find the formulas for the radii of two circles in order that the difference of the areas of the circles shall be d and the sum of their circumferences shall be s.

$$r_1 = \frac{4\,\pi d + s^2}{4\,\pi s} \text{ and } r_2 = \frac{s^2 - 4\,\pi d}{4\,\pi s}. \quad \textit{Ans.}$$

19. Find two fractions whose sum is $\frac{5}{6}$, and whose difference is equal to their product.

20. The diagonal and the longer side of a rectangle are together five times the shorter side, and the longer side exceeds the shorter side by 35 yards. What is the area of the rectangle?

CHAPTER XV

PROGRESSIONS

I. Arithmetic Progression

83. Definition. An ***arithmetic progression*** is a sequence of numbers, called ***terms***, each of which is derived from the preceding by adding to it a fixed amount, called the ***common difference***. An arithmetic progression is denoted by the abbreviation A. P.

Thus 1, 3, 5, 7, $\cdots$ is an A. P. Each term is derived from the preceding by adding 2, which is therefore the common difference. The dots indicate that the sequence may be extended as far as one pleases. Thus the first term after 7 would be 9, the next one would be 11, etc.

Again, 5, 1, -3, -7, -11, $\cdots$ is an A. P. Here the common difference is -4.

EXERCISES

Determine which of the following are arithmetic progressions; determine the common difference and the next two terms of each of the arithmetic progressions.

1. $3, 6, 9, 12, \cdots.$

2. $3, 5, 8, 12, \cdots.$

3. $6, 4, 2, 0, -2, -4, \cdots.$

4. $30, 25, 20, 15, 10, \cdots.$

5. $-1, -1\frac{1}{2}, -2, -2\frac{1}{2}, \cdots.$

6. $a, 2\,a, 4\,a, 5\,a, \cdots.$

7. $a, a+3, a+6, a+9, \cdots.$

8. $a, a+d, a+2\,d, a+3\,d, a+4\,d, \cdots.$

9. $x-4\,y, x-2\,y, x-y, \cdots.$

10. $3\,x+3\,y, 6\,x+2\,y, 9\,x+y, \cdots.$

[Hint. The common difference may always be determined by subtracting any term from the term immediately preceding.]

11. Write the first five terms of the A. P. in which

(*a*) The first term is 5 and the common difference is 2.

(*b*) The first term is -3 and the common difference 1.

(*c*) The first term is $3a$ and the common difference is $-b$.

84. The *n*th Term of an Arithmetic Progression. From the definition (§ 83) it follows that every arithmetic progression is of the form $a, a+d, a+2d, a+3d, a+4d, \cdots$. Here a is the first term and d the common difference.

Observe that the coefficient of d in any one term is 1 less than the number of that term. Thus 2 is the coefficient of d in the *third* term; 3 is the coefficient of d in the *fourth* term, etc. Therefore the coefficient of d in the nth term must be $(n-1)$. Hence, if we let l stand for the nth term, we have the formula

$$\boldsymbol{l=a+(n-1)d.}$$

EXAMPLE. Find the 11th term of the A. P. $1, 3, 5, 7, \cdots$.

SOLUTION. We have $a=1$, $d=2$, $n=11$, $l=?$
The formula gives $l=a+(n-1)d=1+10\times 2=1+20=21$. *Ans.*

EXERCISES

1. Find the 11th term of $3, 6, 9, 12, \cdots$.

2. Find the 13th term of $6, 10, 14, 18, \cdots$.

3. Find the 20th term of $4, 2, 0, -2, -4, \cdots$.

4. Find the 15th term of $-1, -1\frac{1}{2}, -2, -2\frac{1}{2}, \cdots$.

5. Find the 10th term of $x-y, 2x-2y, 3x-3y, \cdots$.

6. When a small heavy body (like a bullet) drops to the ground it passes over 16.1 feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far does it go in the 12th second?

7. If you save 5 cents during the first week in January, 10 cents the second week, 15 cents the third week and so on, how much will you save during the last week of the year?

8. What term of the progression 2, 6, 10, 14, $\cdots$ is equal to 98?

[HINT. $a=2$, $d=4$, $n=?$, $l=98$.]

9. What term of 3, 7, 11, 15, $\cdots$ is equal to 59?

10. The first term of an A. P. is 8 and the 14th term is 47. What is the common difference?

85. The Sum of the First n Terms of an Arithmetic Progression. Let a be the first term of an A. P., d the common difference, l the nth term. Then the *sum* of the first n terms, which we will call S, is

(1) $$S=a+(a+d)+(a+2\,d)+(a+3\,d)+\cdots+(l-d)+l.$$

This value for S may be written in a very much simpler form, as we shall now show.

Write the terms of (1) in their reverse order. This gives

(2) $$S=l+(l-d)+(l-2\,d)+(l-3\,d)+\cdots+(a+d)+a.$$

Now add (1) and (2), noting the cancellation of d with $-d$, $2\,d$ with $-2\,d$, etc. The result is

$$2\,S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l),$$

or $$2\,S=n(a+l).$$

Therefore $$S=\frac{n}{2}(a+l).$$

This is the simple form for S mentioned above. If we replace l by its value $a+(n-1)d$ (§ 84), this result takes the form

$$S=\frac{n}{2}\{2\,a+(n-1)d\}.$$

EXAMPLE. Find the sum of the first 12 terms of the A. P. 2, 6, 10, 14, $\cdots$.

SOLUTION. $a=2$, $d=4$, $n=12$.

Therefore, by the second form for S in § 85, we have

$$S=\tfrac{12}{2}\{4+11\times 4\}=6\{4+44\}=6\times 48=288. \quad \textit{Ans.}$$

EXERCISES

Find the sum of each of the following arithmetic progressions.

1. The first ten terms of 3, 6, 9, 12, $\cdots$.

2. The first fifteen terms of $-2, 0, 2, 4, \cdots$.

3. The first thirteen terms of $1, 3\frac{1}{2}, 6, \cdots$.

4. The first ten terms of $1, -1, -3, -5, \cdots$.

5. The first n terms of 1, 8, 15, $\cdots$.

6. How many strokes does a common clock, striking hours, make in 12 hours?

7. A body falls 16.1 feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far does it fall during the first 12 seconds?

8. Find the sum of all odd integers, beginning with 1 and ending with 99.

9. If you save 5 cents during the first week in January, 10 cents during the second week, 15 cents the third week, and so on, how much will you save in a year?

10. The first term of an A. P. is 4 and the 10th term is 31. What is the sum of the 10 terms?

[HINT. $a=4$, $n=10$, $l=31$. Now use the *first* of the formulas in § 85.]

11. The first term of an A. P. is $\frac{1}{2}$ and the 12th term is $11\frac{1}{2}$. What is the sum of the 12 terms?

12. Figure 49 shows a series of 16 dotted lines which are equally distant from each other. If the highest one is 6 inches long and the lowest one is 3 feet long, what is the sum of all their lengths?

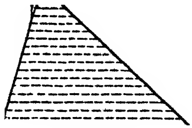


FIG. 49.

[HINT. The lines form an A. P. since their lengths increase uniformly.]

13. The rungs of a ladder diminish uniformly from 2 feet 4 inches long at the base to 1 foot 3 inches long at the top. If there are 24 rungs, what is the total length of wood in them?

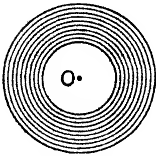


FIG. 50.

14. Find the sum of the circumferences of 10 concentric circles if the radius of the innermost one is $\frac{1}{2}$ inch and the radius of the outermost one is 4 inches, it being understood that the circles are equally spaced from each other.

15. Figure 51 shows a coil of rope in the ordinary circular form, containing 12 complete turns, or layers. If the ength of the innermost turn is 4 inches and the length of the outermost turn is 37 inches, how long is the rope?

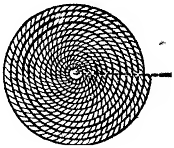


FIG. 51.

[HINT. Regard each turn as a circle, thus neglecting the slight effect due to the overlapping at the beginning of each turn after the first.]

16. If in Fig. 51 the length of the innermost turn is a inches and that of the outermost turn is b inches, and the number of turns is n, what represents the total length of the rope?

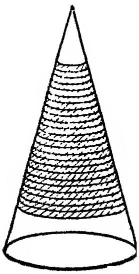


FIG. 52.

17. A small rope is wound tightly round a cone, the number of complete turns being 24. Upon unwinding the rope from the top, the lengths of the first and second turns are found to be $2\frac{1}{2}$ inches and $3\frac{1}{4}$ inches respectively. How long (approximately) is the rope?

86. Arithmetic Means. The terms of an arithmetic progression lying between any two given terms are called the ***arithmetic means*** between those two terms.

Thus, the three arithmetic means between 1 and 9 are 3, 5, 7, since 1, 3, 5, 7, 9 form an arithmetic progression.

Whenever a *single* term is inserted in this way between two numbers, it is briefly called *the arithmetic mean* of those two numbers.

Thus, the arithmetic mean of 2 and 10 is 6, because 2, 6, 10 form an arithmetic progression.

A formula for the arithmetic mean of any two numbers, as a and b, is easily obtained. Thus, if x is the mean, then a, x, b forms an A. P. Therefore, we must have $x-a=b-x$. Solving for x, this gives

$$x=\frac{a+b}{2}.$$

Thus we have the following theorem: *The arithmetic mean of two numbers is equal to half their sum.*

NOTE. The arithmetic mean of two numbers is also called their *average.*

EXAMPLE. Insert five arithmetic means between 3 and 33.

SOLUTION. We are to have an A. P. of 7 terms in which $a=3$, $l=33$, and $n=7$. We begin by finding d. Thus,

$$l=a+(n-1)d \text{ (§ 84) so that } 33=3+6\,d. \quad \text{Solving, } d=5.$$

The progression is therefore 3, 8, 13, 18, 23, 28, 33 and hence the desired means are 8, 13, 18, 23, 28. *Ans.*

EXERCISES

1. Insert three arithmetic means between 7 and 23.
2. Insert four arithmetic means between -5 and 10.
3. Insert seven arithmetic means between $\frac{1}{2}$ and $25\frac{3}{4}$.
4. What is the arithmetic mean of 8 and 30?
5. What is the arithmetic mean of $\frac{1}{2}$ and $-\frac{1}{4}$?

6. Show that the first formula for S obtained in § 85 may be stated as follows: "The sum of n terms of an arithmetic progression is equal to n multiplied by the arithmetic mean of the first and nth terms."

7. $ABCD$ is any *trapezoid* (that is, any four-sided figure having its bases AB and DC parallel to each other). The line EF, called the *median*, joins the middle point of the side AD to the middle point of the side BC, and it is shown in geometry that the length of this line EF will always be the arithmetic mean of the lengths of the bases AB and CD. Hence answer the following questions.

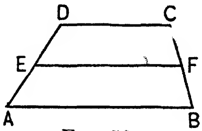


FIG. 53.

(*a*) If the bases are 10 inches long and 2 inches long, respectively, what is the length of the median?

(*b*) If the lower base is 14 inches long and the median 8 inches long, how long is the upper base?

(*c*) If the upper base is 3 feet long and the median 4 feet long, how long is the lower base?

(*d*) If the bases are a inches long and b inches long, respectively, what represents the length of the median?

8. The figure shows the frustum of a cone and the frustum of a pyramid, and in each case the "mid-section" has

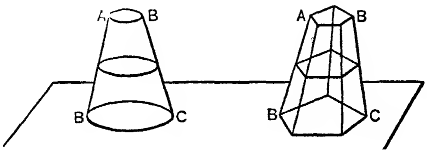


FIG. 54.

been drawn in (that is, the section made by a plane which passes midway between the bases AB and BC). It is shown in solid geometry that in all such cases the perimeter of the

mid-section will always be the arithmetic mean of the perimeters of the two bases. Hence answer the following questions.

(*a*) If the perimeters of the two bases are 30 inches and 10 inches respectively, what is the perimeter of the mid-section?

(*b*) In the frustum of a cone, the radius of the upper base is 2 inches and that of the lower base 8 inches. What is the perimeter of the mid-section?

87. The Five Elements of an Arithmetic Progression. In any arithmetic progression there are the five elements, a, d, l, n, S, defined in §§ 84, 85. If any *three* of these are given, we can always find the other two by means of the formulas in §§ 84, 85.

EXAMPLE 1. Given $a=-\frac{1}{2}$, $n=30$, $S=21\frac{1}{4}$. Find d and l.

SOLUTION. From § 85, we have $S=\frac{n}{2}(a+l)$.

Hence $$21\tfrac{1}{4}=15(-\tfrac{1}{2}+l).$$

Solving, $l=1\frac{11}{12}$. Now, $l=a+(n-1)d$. Hence $1\frac{11}{12}=-\frac{1}{2}+29\,d$.

Solving, $d=\frac{1}{12}$.

EXAMPLE 2. Given $a=3$, $d=4$, $S=300$. Find n and l.

SOLUTION. $S=\frac{n}{2}\{2\,a+(n-1)d\}$. Hence $300=\frac{n}{2}\{6+(n-1)\cdot 4\}$.

Therefore

$$600=n\{4\,n+2\};\ 4\,n^2+2\,n-600=0;\ 2\,n^2+n-300=0.$$

Solving the last (quadratic) equation by formula (§ 56), gives

$$n=\frac{-1\pm\sqrt{1+2400}}{4}=\frac{-1\pm\sqrt{2401}}{4}=\frac{-1\pm 49}{4}=12, \text{ or } -12\tfrac{1}{2}.$$

Since n is the number of terms and therefore a positive integer, it follows that $n=12$. (See Hint to Ex. 3, p. 89.)

To find l, we now use the formula $l=a+(n-1)d$. Thus

$$l=3+11\cdot 4=3+44=47.$$

EXERCISES

1. Given $a=3$, $n=25$, $S=675$, find d and l.

2. Given $a=-9$, $n=23$, $l=57$, find d and S.

3. Given $S=275$, $l=45$, $n=11$, find a and d.

4. Find n and d when $a=-5$, $l=15$, $S=105$.

5. Find a and n when $l=1$, $d=\frac{2}{3}$, $S=-20$.

6. How many terms are there in the arithmetic progression 2, 6, 10, $\cdots$ 70?

7. Given a, l, and n, derive a formula for d.

8. Given a, d, and l, derive a formula for n.

9. Given a, n, and S, derive a formula for l.

10. Given d, l, and S, derive a formula for a.

11. Find an A. P. of 14 terms having 13 for its 6th term and 25 for its 10th term.

12. Find an A. P. of 16 terms such that the sum of the 6th, 7th and 8th terms is $-16\frac{1}{2}$, and the sum of its last two terms is -28.

13. Find three integers in arithmetic progression such that their sum is 24 and their product 384.

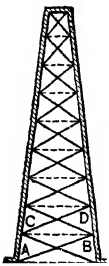


Fig. 55.

14. The figure represents one of the four sides of a steel tower such as is commonly seen at wireless telegraph stations. It is desired to make one of these towers so that each girder, such as AB, will be 2 feet longer than the one just above it, as CD. How many girders will the tower have (counting all four sides) in case the total amount of girder steel used is to be only 864 feet and the lowest girders are each to have a length of 20 feet?

II. Geometric Progression

88. Definitions. A ***geometric progression*** is a sequence of numbers, called ***terms***, each of which is derived from the preceding by multiplying it by a fixed amount, called the ***common ratio***. A geometric progression is denoted by the abbreviation G. P.

Thus 2, 4, 8, 16, 32, $\cdots$ is a G. P. Each term is derived from the preceding by multiplying it by 2, which is therefore the common ratio.

Again, $10, -5, +\frac{5}{2}, -\frac{5}{4}, \cdots$ is a G. P. whose common ratio is $-\frac{1}{2}$. The next two terms are $+\frac{5}{8}, -\frac{5}{16}$.

EXERCISES

Determine which of the following are geometric progressions, and find the common ratio and the next two terms of each geometric progression.

1. $3, 6, 12, 24, 48, \cdots$.

2. $4, 12, 48, 75, \cdots$.

3. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$.

4. $-1, 2, -4, 8, -16,$

5. $a, a^2, a^3, a^4, \cdots$.

6. $2x, 4x^3, 8x^5, 16x^7,$

7. $a, ar, ar^2, ar^3, ar^4, \cdots$.

8. $a, a^2r^2, a^3r^4, a^4r^6, \cdots$.

9. $(a+b), (a+b)^3, (a+b)^5, (a+b)^7, \cdots$.

10. $\frac{m^2}{n^3}, \frac{m^4}{n^4}, \frac{m^6}{n^5}, \frac{m^8}{n^6}, \cdots$.

11. Write the first five terms of the G. P. in which

(*a*) The first term is 4 and the common ratio is 4.

(*b*) The first term is -3 and the common ratio is -2.

(*c*) The first term is a and the common ratio is r.

89. The *n*th Term of a Geometric Progression. From the definition in § 88 it follows that every geometric progression is of the form

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots.$$

Here a is the first term, and r the common ratio.

Observe that the exponent of r in any one term is 1 less than the number of that term. Thus 2 is the exponent of r in the *third* term; 3 is the exponent of r in the *fourth* term, etc. Therefore, the exponent of r in the nth term must be $(n-1)$. Hence, if we let l stand for the nth term, we have the formula

$$\boldsymbol{l = ar^{n-1}}.$$

EXAMPLE. Find the 7th term of the G. P. 6, 4, $\frac{8}{3}$, $\cdots$.

SOLUTION. We have $a = 6$, $r = \frac{2}{3}$, $n = 7$, $l = ?$

The formula gives $l = ar^{n-1} = 6 \times \left(\frac{2}{3}\right)^6 = 2 \times 3 \times \frac{2^6}{3^6} = \frac{2^7}{3^5} = \frac{128}{243}$. *Ans.*

EXERCISES

1. Find the ninth term of 2, 4, 8, 16, $\cdots$.
2. Find the eighth term of $\frac{1}{4}$, $\frac{1}{2}$, 1, $\cdots$.
3. Find the ninth term of -1, 2, -4, 8, $\cdots$.
4. Find the tenth term of 4, 2, 1, $\frac{1}{2}$, $\cdots$.
5. Find the eighth term of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, $\cdots$.
6. Find the eleventh term of ax, a^2x^2, a^3x^3, a^4x^4, $\cdots$.
7. Find the tenth term of 2, $\sqrt{2}$, 1, $\cdots$.
8. What term of the G. P. 3, 6, 12, 24 is equal to 384?
9. What term of the progression 6, 4, $\frac{8}{3}$ is equal to $\frac{64}{81}$?
10. For every person there has lived two parents, four grandparents, eight great grandparents, etc. How many ancestors does a person have belonging to the 7th generation before himself, assuming that there is no duplication? Answer also for the 10th generation.

11. If you save 50 cents during the first three months of the year and double the amount of your savings every three months afterward, how much will you save during the last three months of the second year?

12. From a grain of corn there grew a stalk that produced an ear of 100 grains. These grains were planted and each produced an ear of 100 grains. This was repeated until there were 5 harvestings. If 75 ears of corn make a bushel, how many bushels were there the fifth year?

90. The Sum of the First n Terms of a Geometric Progression. Let a be the first term of a geometric progression, r the common ratio, l the nth term. Then the ***sum*** of the first n terms, which we will call S, is

$$(1) \qquad S=a+ar+ar^2+ar^3+\cdots+ar^{n-2}+ar^{n-1}.$$

This value for S may be written in a very much simpler form, as we shall now show.

Multiply both members of (1) by r. This gives

$$(2) \qquad rS=ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n.$$

Now subtract equation (2) from equation (1), noting the cancelation of terms. This gives

$$S-rS=a-ar^n.$$

Solving this equation for S gives

$$S=\frac{a-ar^n}{1-r}.$$

This is the simple form for S mentioned above.

It is to be observed also that since $l=ar^{n-1}$, we may write $rl=ar^n$. Putting this value of ar^n into the form just found for S, we obtain as a second expression for S the following formula.

$$S=\frac{a-rl}{1-r}.$$

EXAMPLE. Find the sum of the first six terms of the G. P. 3, 6, 9, 12, $\cdots$.

SOLUTION. $a=3, r=2, n=6$. To find S.

$$S=\frac{a-ar^n}{1-r}=\frac{3-3\cdot 2^6}{1-2}=\frac{3-3\cdot 64}{-1}=\frac{3-192}{-1}=\frac{-189}{-1}=189. \quad \textit{Ans.}$$

EXERCISES

Find the sum of the first

1. Eight terms of 2, 4, 8, $\cdots$.
2. Six terms of 1, 5, 25, $\cdots$.
3. Five terms of 1, $1\frac{1}{2}$, $2\frac{1}{4}$, $\cdots$.
4. Six terms of 2, $-\frac{2}{3}$, $\frac{2}{9}$, $\cdots$.
5. Ten terms of $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, $\cdots$.
6. Six terms of 1, $2a$, $4a^2$, $\cdots$.
7. Ten terms of 1, a^2, a^4, $\cdots$.
8. What is the sum of the series 3, 6, 12, $\cdots$, 384?
9. What is the sum of the series 8, 4, 2, $\cdots$, $\frac{1}{16}$?
10. Find the sum of the first ten powers of 2.
11. Find the sum of the first seven powers of 3.
12. A series of five squares are drawn such that a side of the second one is twice as long as a side of the first one, a side of the third one is twice as long as a side of the second, etc. If a side of the first one is 2 inches long, find (by § 90) the sum of the areas of all the squares.
13. What is the combined volume of five spheres if the radius of the first one is 16 inches, the radius of the second one is half that of the first one, the radius of the third one is half that of the second one, and so on to the fifth one? (See Ex. 14 (e), p. 6.)
14. Half the air in a certain corked empty jug is removed by each stroke of an air pump. What fraction of the original volume of air has been removed by the end of the seventh stroke?

Historical Note. It is related that when Sessa, the inventor of chess, presented his game to Scheran, an Indian prince, the latter asked him to name his reward. Sessa begged that the prince would give him 1 grain of wheat for the first square of the chess board, 2 for the second, 4 for the third, 8 for the fourth, and so on to the sixty-fourth. Delighted with the inventor's modesty, the prince ordered his ministers to make immediate payment. The number of grains of wheat thus called for was (see § 90)

$$\frac{1-1\cdot 2^{64}}{1-2}=\frac{2^{64}-1}{1}=2^{64}-1.$$

But the value of 2^{64} is the enormous number 18,446,744,073,709, 551,616, so the number of grains of wheat owing was but 1 less than this. This amount is greater than the world's annual supply at present. History does not relate how the claim was settled. (From Godfrey and Siddons' *Elementary Algebra*, Vol. II, pp. 336, 337.)

91. Geometric Means. The terms of a geometric progression lying between any two given terms are called the ***geometric means*** of those two terms.

Thus the three geometric means of 2 and 32 are 4, 8, 16, since 2, 4, 8, 16, 32 form a geometric progression.

Whenever a *single* term is inserted in this way between two numbers, it is briefly called the *geometric mean* of those two numbers.

Thus the geometric mean of 2 and 32 is 8, since 2, 8, 32 form a geometric progression.

A formula for the geometric mean of any two numbers, as a and b, is easily obtained. Thus, if x is the mean, then a, x, b forms a G. P. Therefore we must have $x/a=b/x$. Solving, we have $x^2=ab$, and hence

$$x=\sqrt{ab}.$$

Thus, we have the following theorem: *The geometric mean of two numbers is equal to the square root of their product.*

Note. The geometric mean of two numbers is thus the same as their mean proportional. See Ex. 6, p. 80.

EXAMPLE. Insert four geometric means between 3 and 96.

SOLUTION. We are to have a G. P. of six terms in which $a = 3$, $l = 96$, and $n = 6$. We begin by finding r. Thus

$$l = ar^{n-1} (\S\ 89) \text{ so that } 96 = 3 \cdot r^5, \text{ or } r^5 = 32. \quad \text{Hence } r = 2.$$

The progression is therefore 3, 6, 12, 24, 48, 96, and hence the desired means are 6, 12, 24, 48. *Ans.*

EXERCISES

1. Insert four geometric means between 2 and 486.
2. Insert three geometric means between 1 and 625.
3. Insert five geometric means between $4\frac{1}{2}$ and $\frac{1}{162}$.
4. What is the geometric mean of 2 and 18?
5. What is the geometric mean of 8 and 50?
6. What is the geometric mean of $\frac{1}{2}$ and $3\frac{5}{9}$.
7. Find, correct to four decimal places, the geometric mean of 6 and 27, using the tables of square roots.
8. Find, correct to four decimal places, the geometric mean of $2\frac{1}{2}$ and $3\frac{1}{4}$.
9. Insert two geometric means between 5 and 9, expressing each correct to four decimal places.
10. Show that the number of units in a side of the square is the geometric mean of the number of units in the two unequal sides of a rectangle that has the same area.
11. Figure 56 shows a square within which is placed (in any manner) another square whose side is half as long as that of the first square. Show that the area between the squares is equal to three halves of the mean proportional between the areas of the squares themselves.

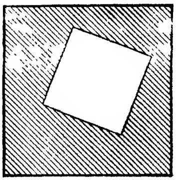


FIG. 56.

12. Show that the result stated in Ex. 11 holds true also in the case of the area between two circles, the smaller circle lying within the larger and having its radius half as long as that of the larger circle. Draw a figure.

92. Infinite Geometric Progression. Consider the geometric progression

(1) $$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots.$$

Here $a=1$, $r=\frac{1}{2}$, and hence, by § 90, the sum of n terms is

$$S=\frac{a-ar^n}{1-r}=\frac{1-1\cdot(\frac{1}{2})^n}{1-\frac{1}{2}}=\frac{1-(\frac{1}{2})^n}{\frac{1}{2}}.$$

Now, if the value selected for n is very large, the expression $(1/2)^n$, which here appears, is very small, being the fraction $\frac{1}{2}$ multiplied into itself n times. In fact, as n is selected larger and larger, this expression $(1/2)^n$ comes to be as small as we please, so that the value for S, as given above, comes as near as we please to

$$\frac{1-0}{\frac{1}{2}},$$

which is the same as 2. So we say that 2 is the *sum to infinity* of the geometric progression above, meaning thereby simply that as we sum up the terms, taking more and more of them, we come *as near as we please* to 2.

The meaning of this result is seen in the figure below.

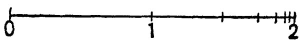


FIG. 57.

Here, beginning at the point marked 0, we first measure off 1 unit of length, then, continuing to the right, we measure off $\frac{1}{2}$ unit, then $\frac{1}{4}$ unit, then $\frac{1}{8}$ unit, etc., each time going to the right just one half the amount we went the time before. As this is kept up *indefinitely*, we evidently come as near as we please to the point marked 2, which is 2 units from 0. This corresponds exactly to what we are doing when we add more and more of the terms of the given progression

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots.$$

A progression like the one just considered, in which the value of n is not stated but may be taken as large as one pleases, is called an *infinite* geometric progression.

Having thus considered the sum to infinity of the *special* infinite geometric progression (1), let us now suppose that we have *any* infinite geometric progression, as

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,$$

and (as before) that r has some value numerically (§ 1) *less than* 1. Then the sum of the first n terms is

$$S=\frac{a-ar^n}{1-r},$$

and, as n is taken larger and larger, the expression r^n which appears here becomes as small as we please, since we have supposed r to be less than 1. Hence, as n increases indefinitely, the value of S comes as near as we please to

$$\frac{a-a\cdot 0}{1-r},$$

or

$$\frac{a}{1-r}.$$

We have therefore the following theorem: *The sum to infinity of any geometric progression whose common ratio r is numerically less than* 1 *is given by the formula*

$$S=\frac{a}{1-r}.$$

EXAMPLE. Find the sum to infinity of the progression

$$3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \tfrac{1}{27},\ \cdots.$$

SOLUTION. $a=3$, $r=\frac{1}{3}$. Since r is numerically less than 1, we have by the formula of § 92,

$$S=\frac{a}{1-r}=\frac{3}{1-\frac{1}{3}}=\frac{3}{\frac{2}{3}}=\frac{9}{2}=4\tfrac{1}{2}. \quad \textit{Ans.}$$

EXERCISES

Find the sum to infinity of each of the following progressions, and state in each case what your answer *means*, drawing a diagram similar to Fig. 57 to illustrate.

1. $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.
2. $3, \frac{8}{4}, \frac{3}{16}, \frac{3}{64}, \cdots$.
3. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \cdots$.

[HINT. $r=-\frac{1}{3}$ and hence is numerically less than 1. The formula of § 92 therefore applies.]

4. $5, .5, .05, .005, \cdots$.
5. $\frac{1}{8}, -\frac{1}{18}, \frac{2}{81}, \cdots$.
6. $1-x+x^2-x^3+\cdots$ when $x=\frac{2}{3}$.
7. $\sqrt{2}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \cdots$.
8. $\frac{2}{3}, -\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{4}{9}, \cdots$.
9. $\frac{4}{5}, \frac{2}{5\sqrt{3}}, \frac{1}{15}, \cdots$.

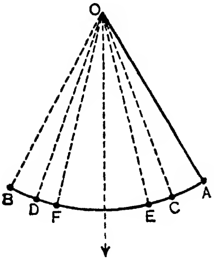


FIG. 58.

10. A pendulum starts at A and swings to B, then it swings back as far as C, then forward as far as D, etc. If the first swing (that is, the circular arc from A to B) is 6 inches long and each succeeding swing is five sixths as long as the one just preceding it, how far will the pendulum bob travel before coming to rest?

11. At what time after 3 o'clock do the hands of a watch pass each other?

[HINT. We may look at this as follows; The large (minute) hand first moves down to where the small (hour) hand is at the be-

ginning, that is through 15 of the minute spaces along the dial. Meanwhile the small hand advances $\frac{1}{12}$ as far or $\frac{15}{12}$ of a minute space. This brings the small hand to the position indicated by the dotted line in the figure. The large hand next passes over this $\frac{15}{12}$ of a minute space. Meanwhile the small hand again advances $\frac{1}{12}$ as far, which is $\frac{15}{144}$ of a minute space. The large hand next covers this $\frac{15}{144}$ of a minute space, but the small hand meanwhile advances $\frac{1}{12}$ as far, or $\frac{15}{1728}$ of a minute space, etc. Thus, the successive moves of the large hand, counting from the first one, form the G. P. 15, $\frac{15}{12}$, $\frac{15}{144}$, $\frac{15}{1728}$, $\cdots$. The sum of this to infinity will be the total distance passed over by the large hand before the hands pass.]

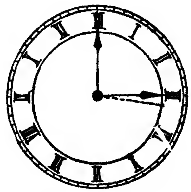


FIG. 59.

93. Variable. Limit. We have seen (§ 92), in connection with the geometric progression 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$, that the sum of its first n terms is a quantity which, as n increases indefinitely, comes and remains as near as we please to the exact value 2. The usual way of stating this is to say that *as n increases, the sum of the first n terms approaches* 2 *as a* ***limit***. The sum of the first n terms is here called a ***variable*** since it varies, or changes, in the discussion. A similar remark applies to all the infinite geometric progressions which we have considered. In every case the sum to infinity is the limit which the sum of the first n terms, considered as a variable quantity, is approaching.

NOTE. It may be asked whether the sum of the first n terms of the G. P. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$ could ever actually reach its limit 2. The answer is that it may or it may not, depending upon circumstances. Thus, if we think of the terms, beginning with the second, as being added on at the rate of one a minute we could never reach the end of the adding process, since the number of the terms is inexhaustible and hence the minutes required would have no end. In other words, the sum of the first n terms could never reach its limit on this plan.

But suppose that instead of this we were to add on the terms with increasing speed as we went forward. For example, suppose we added on the $\frac{1}{2}$ in $\frac{1}{2}$ a minute, then the $\frac{1}{4}$ in $\frac{1}{4}$ of a minute, then the $\frac{1}{8}$ in $\frac{1}{8}$ of a minute, etc. On this plan we would actually reach the limit 2 in 2 minutes of time. Here the constantly increasing speed of the adding process exactly counterbalances the fact that we have an indefinitely large number of terms to add, with the result that we reach the end of the process in the definite time of 2 minutes. This idea is practically illustrated in Ex. 11, p. 159, where the hands of the watch would never pass each other at all except for the fact that the successive moves of the large hand, which constitute the terms of the progression $15, \frac{15}{12}, \frac{15}{144}, \frac{15}{1728}, \cdots$ are added on in less and less time as the process goes on, each being added on in $\frac{1}{12}$ the time occupied by the one just before it.

The question of whether a variable can reach its limit is intimately connected with the famous problem considered by the Schoolmen in the Middle Ages and known as the problem of Achilles and the tortoise. In this problem, Achilles, who was a celebrated runner and athlete, starts out from some point, as A, to overtake a tortoise which is at some point, as T, the tortoise being famous for the slow rate at which it crawls along. Both start at the same instant and go in the same direction, as indicated in the figure.

Fig. 60.

Achilles soon arrives at the point T, from which the tortoise started, but in the meantime the tortoise has gone some distance ahead. Achilles now covers this last distance, but this leaves the tortoise still ahead, having again gained some additional distance. This continues indefinitely. *How, therefore, can Achilles ever overtake the tortoise?* The Schoolmen never quite answered this question satisfactorily to themselves. The secret of the difficulty lies in the fact that, as in the other problems mentioned above, the successive moves which Achilles makes are done in shorter and shorter intervals of *time*, with the result that, although the number of moves necessary is indefinitely great, they can all be accomplished in a definite time.

94. Repeating Decimals. If we express the fraction $\frac{12}{33}$ decimally by dividing 12 by 33 in the usual way, we find that the quotient is .363636 ⋯, the dots indicating that the division process never stops (or is never exact) but leads to a never-ending decimal. However, the figures appearing in this decimal are seen to repeat themselves in a regular order, since they are made up of 36 repeated again and again. Such a decimal is called a *repeating decimal.* More generally, a repeating decimal is one in which the figures repeat themselves *after a certain point.* Thus, .12343434 ⋯, 1.653653653 ⋯, are repeating decimals.

Let us now turn the question around. Thus, suppose that a certain repeating decimal is given, as for example .272727 ⋯, and let us ask what fraction when divided out gives this decimal. This kind of question is usually too difficult to answer in arithmetic, but it can be easily answered as follows by use of the formula in § 92.

Thus the decimal .272727 ⋯ may be written in the form

$$\tfrac{27}{100}+\tfrac{27}{10000}+\tfrac{27}{1000000}+\cdots.$$

This is an infinite geometric progression in which $a=\frac{27}{100}$, $r=\frac{1}{100}$. The sum of this progression to infinity must be the value of the given decimal. Hence, the desired value is

$$\frac{a}{1-r}=\frac{\frac{27}{100}}{1-\frac{1}{100}}=\frac{27}{100}\times\frac{100}{99}=\frac{27}{99}=\frac{3}{11}. \quad \textit{Ans.}$$

This answer may be checked by dividing 3 by 11, the result being .272727 ⋯, which is the given decimal.

NOTE. It is shown in higher mathematics that every rational fraction in its lowest terms (that is, every number of the form a/b, where a and b are integers prime to each other) gives rise when divided out to a never-ending, *repeating* decimal, while every irrational number (such as $\sqrt{2}$) gives rise when expressed decimally to a never-ending *non-repeating* decimal.

EXERCISES

Find the values of the following repeating decimals and check your answer for each of the first six.

1. .414141 ⋯. **2.** .898989 ⋯. **3.** .543543543 ⋯.

4. .3414141 ⋯.

SOLUTION.
$$\begin{aligned}.3414141\cdots &= .3+.0414141\cdots\\ &= .3+\tfrac{1}{10}(.414141\cdots)\\ &= .3+\frac{1}{10}\left(\frac{\frac{41}{100}}{1-\frac{1}{100}}\right)\\ &= \frac{3}{10}+\frac{1}{10}\times\frac{41}{100}\times\frac{100}{99}\\ &= \frac{3}{10}+\frac{41}{990}=\frac{338}{990}=\frac{169}{495}.\quad \textit{Ans.}\end{aligned}$$

5. .6535353 ⋯.

6. 1.212121 ⋯.

7. 3.2151515 ⋯.

8. 5.032032032 ⋯.

9. 6.008008008 ⋯.

10. 34.5767676 ⋯.

MISCELLANEOUS PROBLEMS

I. ARITHMETIC PROGRESSION

1. What will be the cost of digging a 20-foot well if the digging costs 50 cents for the first foot and increases by 25 cents for each succeeding foot?

2. Fifty-five logs are to be piled so that the top layer shall consist of 1 log, the next layer of 2 logs, the next layer of 3 logs, etc. How many logs will lie on the bottom layer?

3. In a potato race 30 potatoes are placed at the distances 6 feet, 9 feet, 12 feet, etc., from a basket. A player starts from the basket, picks up the potatoes and carries them, one at a time, to the basket. How far does he go altogether in doing this?

4. A row of numbers in arithmetic progression is written down and afterwards all erased except the 7th and the 12th, which are found to be -10 and 15, respectively. What was the 20th number?

5. If your father gives you as many dimes on each of your birthdays as you are years old on that day, how old will you be when the total amount he has given you in this way amounts to $12?

6. How many arithmetic means must be inserted between the numbers 4 and 25 in order that their sum may amount to 87?

7. Prove that equal multiples of the terms of an arithmetic progression are in arithmetic progression.

8. Prove that the sum of n consecutive odd integers, beginning with 1, is n .

9. The sum of three numbers in arithmetic progression is 30 and the sum of their squares is 462. What are the numbers?

[Hint. The numbers may be represented as $x-y$, x, $x+y$. Form two equations and solve for x and y.]

10. If a person saves $20 the first month and $10 each month thereafter, how long before his total savings will amount to $1700?

11. Divide 80 into four parts which are in arithmetic progression and which are such that the product of the first and fourth is to the product of the second and third as $2:3$.

12. Find the sum of the first 40 terms of an A.P. in which the ninth term is 136 and the sum of the first nineteen terms is 2527.

13. If $d=2$, $n=21$, and $S=147$, find a and l.

14. Show that if, in any A.P., the values of d, l, and S are given, then the formula for a is

$$a=\tfrac{1}{2}d\pm\sqrt{(l+\tfrac{1}{2}d)^2-2dS}.$$

II. Geometric Progression

15. A wheel in a certain piece of machinery is making 32 revolutions per second when the steam is turned off and the wheel begins to slow down, making one half as many revolutions each second as it did the preceding second. How long before it will be making only 2 revolutions per second?

16. Show that if a principal of $\$p$ be invested at $r\ \%$ compound interest, the sum of money accumulating at the ends of successive years will form a geometric progression, while if the investment be made at simple interest, the sums accumulating will form an arithmetic progression.

17. From a cask of vinegar $\frac{1}{3}$ the contents is drawn off and the cask then filled by pouring in water. Show that if this is done 6 times, the cask will then contain more than 90% water.

[Hint. Call the original amount of vinegar 1, then express (as a proper fraction) the amount of water in the cask after the first refilling, second refilling, etc.]

18. A set of concentric circles is drawn, each having a radius half that of the circle just outside it. Show that the limit toward which the sum of their circumferences is approaching is equal to twice the circumference of the largest circle.

19. A dipper when hung on a wall often swings back and forth for a time, the swings gradually dying out. If the first swing occupies 1 second, and each succeeding swing takes .9 as long as the one before it, how long before the dipper comes to rest?

20. It is found by experiment that the number of bacteria in a sample of milk doubles every 3 hours. What increase will there be in 24 hours, assuming that all outside conditions remain the same?

21. In Fig. 61 a series of ordinates equally spaced from each other has been drawn, the first one being laid off 1 unit long, the second one being laid off equal to the first one increased by $\frac{1}{4}$ its length, the third being equal to the second increased by $\frac{1}{4}$ its length, etc. Show that these ordinates represent the successive terms of the G. P. whose first term is 1 and whose common ratio is $1\frac{1}{4}$. In this sense, the figure may be called the diagram for the G. P. in which $a=1$, $r=1\frac{1}{4}$.

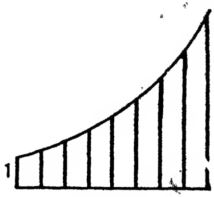


FIG. 61.

22. Draw the diagram for the G. P. in which

(a) $a=1, r=1\frac{1}{3}$; (b) $a=2, r=1\frac{1}{6}$; (c) $a=4, r=\frac{1}{2}$.

[HINT. Use 8 ordinates only, spacing them at any convenient but equal distance apart.]

23. Prove that any series of numbers formed by writing down the reciprocals of the successive terms of a geometric progression is itself a geometric progression.

24. Three numbers whose sum is 24 are in arithmetic progression, but if 3, 4, and 7 be added to them respectively, the results form a geometric progression. Find the numbers.

25. If a series of numbers are in G. P., are their squares likewise in G. P.? Answer the same for their cubes; also for their square roots and their cube roots.

Answer the same questions for an A. P.

[HINT. See that your reasoning is *general*, that is, do not base it upon an examination of some special cases.]

CHAPTER XVI

RATIO AND PROPORTION

95. Ratio. The quotient of one number divided by another of the same kind is called their ***ratio***.

Thus the ratio of 6 inches to 3 inches is $\frac{6}{3}$, or 2; the ratio of 5 lb. to 3 lb. is $\frac{5}{3}$, etc. Note that in each of these cases the ratio is simply a fraction of the kind studied in arithmetic.

The first number, or dividend, is called the ***antecedent;*** the second number, or divisor, is called the ***consequent.***

Thus, in the ratio $\frac{3}{4}$, the antecedent is 3 and the consequent is 4.

EXERCISES

1. What is the ratio of 10 yards to 2 yards? of 7 yards to 3 yards?

2. State (as a fraction in its simplest form) the value of each of the following ratios.

(*a*) 5 to 25. (*c*) $\frac{1}{2}$ to $\frac{1}{3}$. (*e*) 1 to 3. (*g*) $18\,x^2$ to $4\,x^2$.
(*b*) 16 to 12. (*d*) 2 to $\frac{1}{2}$. (*f*) $3\,a$ to $6\,b$. (*h*) x^2-y^2 to $x-y$.

3. State which is the antecedent and which the consequent in each of the parts of Ex. 2.

4. What is the ratio of 10 inches to 2 feet?

[HINT. First reduce the 2 feet to inches so that we may compare *like* numbers, that is numbers measured in the *same* unit.]

5. The dimensions of a certain grain bin are 3 feet by 6 feet by 7 feet. What is the ratio of its cubical contents to that of a bin whose dimensions are 3 feet 6 inches by 5 feet by $1\frac{1}{2}$ yards?

6. If one square has its sides each twice as long as the sides of another square, what is the ratio of the area of the first square to that of the second?

[HINT. Let a = a side of the smaller square.]

7. If one cube has its edges each twice as long as the edges of another cube, what is the ratio of the volume of the first cube to that of the second?

8. Show that if a cylinder and a cone have the same circular base and the same height, the ratio of the volume of the cylinder to that of the cone is $3:1$.

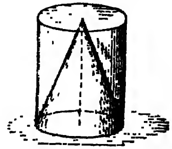


FIG. 62.

9. When we sharpen a lead pencil a certain part of the cylindrical lead is exposed. What part of the exposed lead is cut off when a smooth conical point is made?

96. Proportion. A ***proportion*** is an expression of equality between two ratios, or fractions.

For example, since $\frac{1}{2}$ is the same as $\frac{2}{4}$, we have the proportion $\frac{1}{2}=\frac{2}{4}$. Likewise, we may write $\frac{2}{3}=\frac{4}{6}$, $\frac{6}{7}=\frac{18}{21}$, $-\frac{2}{3}= -\frac{6}{9}$, etc.; hence all these are true proportions. But $\frac{2}{3}=\frac{1}{3}$ is *not* a proportion since these two fractions are *unequal.*

Every proportion is thus seen to be an equality of the form $a/b=c/d$, where a, b, c, and d are certain numbers. These four numbers are called the ***terms*** of the proportion. The first and fourth (that is, a and d) are called the ***extremes*** of the proportion, while the second and third (b and c) are called the ***means.***

Besides writing a proportion in the form $a/b=c/d$, it may be written in the form $a:b::c:d$, or also in the form $a:d= c:d$. In all cases it is read "*a is to b as c is to d,*" and it means that the fraction a/b equals the fraction c/d.

EXERCISES

1. Using the language of proportion, read each of the following statements.

(a) $\frac{3}{4}=\frac{6}{8}$.
(b) $1:4=3:12$.
(c) $2:-1=8:-4$.
(d) $\frac{1}{3}:\frac{1}{4}::4:3$.

2. State which are the extremes and which the means in each part of Ex. 1.

3. State such proportions as you can make out of the following four quantities: 3 inches, 6 inches, 12 inches, 24 inches.

[HINT. 3 inches is to 6 inches as : · · . Make other proportions also.]

4. State such proportions as you can make out of the following four quantities: 1 inch, 3 inches, 1 foot, 1 yard.

[HINT. First express all quantities in inches.]

5. State such proportions as you can make out of the following: 1 pint, 1 quart, 1 gallon, 2 gallons.

6. Do as in Ex. 5 for the following: 2 seconds, 1 minute, 1 hour, a day and a quarter.

7. Do as in Ex. 5 for the following: 1 cent, 1 dollar, 1 centimeter, 1 meter.

[HINT. Compare money ratio with distance ratio.]

8. Do as in Ex. 5 for the following: 4 ounces, 1 pound, 1 gallon, 1 quart.

97. Algebraic Proportions. If we consider the algebraic fraction $(a^2b)/(ab^2)$, we see (upon dividing both numerator and denominator by ab) that it reduces to a/b. In other words, we have

$$\frac{a^2b}{ab^2}=\frac{a}{b}.$$

This is an example of an *algebraic proportion.* Similarly,

$$\frac{2\,x^2y}{4\,xyz}=\frac{x}{2\,z}$$

is an algebraic proportion and may be written if desired in the form

$$2\,x^2y : 4\,xyz = x : 2\,z.$$

Likewise, since

$$\frac{a^2-b^2}{a-b} = a+b = \frac{a+b}{1},$$

we have

$$a^2-b^2 : a-b = a+b : 1.$$

98. Fundamental Theorem. Let $a/b = c/d$ be any proportion. By multiplying both sides of this equality by bd, we obtain

$$\frac{a}{\cancel{b}} \times \cancel{b}d = \frac{c}{\cancel{d}} \times b\cancel{d},$$

or

$$ad = bc.$$

This result may be stated in the following theorem.

THEOREM A. *In any proportion, the product of the means is equal to the product of the extremes.*

This theorem is useful in testing the correctness of a proportion. Thus $6 : 9 = 14 : 21$ is a correct proportion because the product of the means, which is 9×14, is equal to the product of the extremes, which is 6×21; but $6 : 9 = 8 : 15$ is *not* correct because 9×8 is *not* equal to 6×15. Similarly, $x^3 : x^2y = x : y$, because $x^2y \cdot x = x^3 \cdot y$.

EXERCISES

By means of the theorem of § 98, test the correctness of the following proportions.

1. $5 : 6 = 15 : 18.$
2. $3 : 2 = 5 : 6.$
3. $4 : -1 = 8 : -2.$
4. $\frac{1}{2} : \frac{1}{4} = 8 : 4.$
5. $17 : 19 = 21 : 23.$
6. $2\,a : ab = 10\,x : 5\,bx.$
7. $3\,m^2 : (a-b) = 6\,m : 2\,m(a-b).$
8. $(x^2-y^2) : (2\,x+2\,y) = (2\,x-2\,y) : 4.$
9. $(a^2-b^2) : (a+b)^2 = (a-b) : (a+b).$

By means of the theorem of § 98, determine the value which x must have in the following proportions.

10. $x:4=3:2$.

[HINT. The theorem gives $4\cdot 3=x\cdot 2$, or $12=2x$.]

11. $10:x=2:5$.

12. $25/32=8/x$.

13. $(x-5):4::2:3$.

14. $(x-3)/(x-4)=5/6$.

15. What number bears the same ratio to 4 as 16 does to 6?

[HINT. Let x represent the unknown number and form a proportion.]

16. Divide 35 into two parts whose ratio shall be $\frac{2}{3}$.

[HINT. Let x be one part. Then $35-x$ will be the other part.]

17. Divide 35 into two parts such that the lesser diminished by 4 is to the larger increased by 9 as $1:3$.

18. A man's income from two investments is $980. The two investments bear interest rates which are in the ratio of 5 to 6. What income does he receive from each?

19. Concrete for sidewalks is a mixture made of two parts sand to one part cement. How much of each is required to make a walk containing 1500 cubic feet?

20. Prove that no four consecutive numbers, as n, $n+1$, $n+2$, $n+3$, can form a proportion in the order given.

99. Application to Similar Figures. When two geometric figures have the same shape, though not necessarily the same size, they are called ***similar figures***. Thus any two circles are similar figures; likewise, any two squares, or any two cubes, or any two spheres.

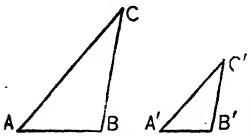


FIG. 63.

Two triangles may be similar, as illustrated in Fig. 63.

The following facts are shown in geometry regarding any two similar figures.

(*a*) *Corresponding lines are proportional.*

Thus, in the two similar triangles of Fig. 63, if the side AB of the one is twice as long as the corresponding side $A'B'$ of the other, then BC is twice as long as $B'C'$. That is,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

In the same way, we have also

$$\frac{AB}{A'B'}=\frac{CA}{C'A'}.$$

(*b*) *Areas are proportional to the* **squares** *of corresponding lines.*

Thus, if one circle has a radius of length R and another circle has a radius of length r, the area, A, of the first is to the area, a, of the second as R^2 is to r^2. That is we have the proportion $A/a=R^2/r^2$.

(*c*) *Volumes are proportional to the* **cubes** *of corresponding lines.*

Thus, if one sphere has the radius R and another has the radius r, the volumes V and v of the spheres are such that $V/v=R^3/r^3$.

EXERCISES ON SIMILAR FIGURES

1. In the two similar triangles shown in § 99 suppose $AB=2$ feet, $A'B'=1$ foot 4 inches, and $BC=3$ feet. How long will $B'C'$ be?

2. If a tree casts a shadow 40 feet long when a post $3\frac{1}{4}$ feet high casts a shadow 4 feet long, how high is the tree?

3. Compare the areas of two city lots of the same shape if a side of the one is three times as long as the corresponding side of the other. Does it matter what the shape of each is?

4. If a certain bottle holds $\frac{1}{2}$ pint, how much will a bottle of the same shape but only half as high hold?

100. Mean Proportional. If the means of a proportion are equal, either is called the ***mean proportional*** between the extremes.

Thus 2 is the mean proportional between 1 and 4 because $\frac{1}{2}=\frac{2}{4}$. Likewise, $2x$ is the mean proportional between x^2 and 4, because $x^2/2x=2x/4$.

NOTE. The mean proportional between a and b is always equal to $\sqrt{ab}$, for we must have $a/x=x/b$. Hence, clearing of fractions, $x^2=ab$, and therefore $x=\sqrt{ab}$. This will be a surd (§ 42) unless the product ab is a perfect square. For example, the mean proportional between 2 and 3 is the surd $\sqrt{2\cdot 3}$, or $\sqrt{6}=2.44949$ (table).

101. Third and Fourth Proportionals. The ***third proportional*** to two numbers a and b is that number x such that $a:b=b:x$.

Thus the third proportional to 2 and 3 is the value of x in the equation
$$\frac{2}{3}=\frac{3}{x}.\quad \text{Solving, } x=\tfrac{9}{2}=4\tfrac{1}{2}.\quad Ans.$$

The ***fourth proportional*** to three numbers a, b, and c is that number x such that $a:b=c:x$.

Thus the fourth proportional to 2, 3, and 4 is the value of x in the equation $2/3=4/x$. Solving, $x=6$. *Ans.*

EXERCISES

1. Find the mean proportional between 8 and 18.

[HINT. Let x be the desired mean. Then $8/x=x/18$. Solve for x.]

Find the mean proportional between each of the following pairs of numbers. In cases where the answer is a numerical surd, use the table to find its approximate value.

2. 9 and 81.
3. 6 and 7.
4. 5 and 20.
5. 5 and 19.
6. $\frac{4}{9}$ and $\frac{16}{25}$.
7. $2\frac{1}{2}$ and $3\frac{1}{2}$.
8. $2x^2y$ and $32xy^2$.
9. $\dfrac{a^2-5a+6}{a+1}$ and $\dfrac{a^2-2a-3}{a-3}$.

Find the third proportional to each of the following pairs.

10. 3 and 4. **12.** $2\frac{1}{2}$ and $3\frac{1}{2}$. **14.** x^2-9 and $x-3$.

11. 18 and 50. **13.** $2x$ and x. **15.** 2 and 6.

Find the fourth proportional to each of the following sets.

16. 3, 4, and 5. **19.** $\sqrt{2}$, $\sqrt{6}$, and $\sqrt{12}$.

17. 5, 4, and 2. **20.** $3a$, $2b$, and c.

18. 2, $3\frac{1}{2}$, $4\frac{1}{3}$. **21.** x, y, and xy.

NOTE. In Exs. 16–21, the numbers must be placed in the proportion in the order in which they are given, as in the illustrative examples of § 101.

22. In the semicircle ABC suppose DE drawn perpendicular to AB. Then (as shown in geometry) the length of DE will be the mean proportional between the lengths of AE and EB.

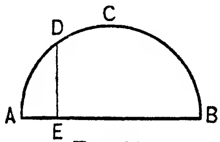


FIG. 64.

If $AE=4$ inches and $EB=16$ inches, find DE.

23. The figure shows a circle and a point P outside it from which are drawn two lines PS and PT. The first of these lines (called a *secant*) cuts through the circle at two points R and S while the second line (called a *tangent*) just *touches* the circle at the point T. In all such cases, the tangent length, PT, is the mean proportional between the whole secant, PS, and its external segment, PR (as shown in geometry).

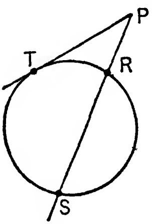


FIG. 65.

Find the length of PT if $PR=4$ and $RS=11$.

24. If a, b, c, d are unequal numbers such that $a:b=c:d$, show that no number x can be found such that

$$a+x:b+x=c+x:d+x.$$

102. Second Fundamental Theorem. THEOREM B. *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion in which the other two are taken as the extremes.*

PROOF. Suppose $mn = xy$. Dividing both members by nx gives $m/x = y/n$, or $m : x = y : n$, which is one of the possible proportions mentioned in the theorem.

Similarly, if we divide both members of $mn = xy$ by ny we obtain $m/y = x/n$, or $m : y = x : n$, which is another of the possible proportions mentioned in the theorem.

The other possible proportions are $x : m = n : y$ and $n : x = y : m$. The proof of these is left to the pupil.

For example, the equality $2 \cdot 9 = 3 \cdot 6$ gives rise to the proportions $2 : 3 = 6 : 9$, $2 : 6 = 3 : 9$, $9 : 3 = 6 : 2$, and $9 : 6 = 3 : 2$.

103. Inversion in a Proportion. THEOREM C. *If four quantities are in proportion, they are in proportion by* ***inversion;*** that is the second term is to the first as the fourth is to the third.

PROOF. We are to show that if $a/b = c/d$, then $b/a = d/c$.

Since $a/b = c/d$, we have, by Theorem A, $ad = bc$.

Therefore, by Theorem B, we may write $b/a = d/c$.

For example, $\frac{2}{3} = \frac{6}{9}$ gives *by inversion* the new proportion $\frac{3}{2} = \frac{9}{6}$.

104. Alternation in a Proportion. THEOREM D. *If four quantities are in proportion, they are in proportion by* ***alternation;*** that is the first term is to the third as the second is to the fourth.

The proof is left to the pupil. First use Theorem A. See proof of Theorem C.

For example, $\frac{2}{3} = \frac{6}{9}$ gives *by alternation* the new proportion $\frac{2}{6} = \frac{3}{9}$.

105. Composition in a Proportion. THEOREM E. *If four quantities are in proportion, they are in proportion by*

composition; that is the sum of the first two terms is to the second term as the sum of the last two terms is to the last term.

PROOF. We are to show that if $a/b=c/d$, then $(a+b)/b=(c+d)/d$.

Since $a/b=c/d$, we may add 1 to each member of this equation, thus giving

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ which reduces to } \frac{a+b}{b}=\frac{c+d}{d}.$$

For example, $\frac{2}{3}=\frac{6}{9}$ gives *by composition* the new proportion $(2+3)/3=(6+9)/9$, or $\frac{5}{3}=\frac{15}{9}$.

106. Division in a Proportion. THEOREM F. *If four quantities are in proportion, they are in proportion by* ***division;*** that is, the difference between the first two terms is to the second term as the difference between the last two terms is to the last term.

PROOF. We are to show that if $a/b=c/d$, then $(a-b)/b=(c-d)/d$.

Since $a/b=c/d$, we may subtract 1 from each side of this equation, thus obtaining

$$\frac{a}{b}-1=\frac{c}{d}-1, \text{ which reduces to } \frac{a-b}{b}=\frac{c-d}{d}.$$

For example, $\frac{3}{2}=\frac{9}{6}$ gives *by division* the new proportion $(3-2)/2=(9-6)/6$, or $\frac{1}{2}=\frac{3}{6}$.

107. Composition and Division. THEOREM G. *If four quantities are in proportion, they are in proportion by* ***composition and division;*** that is the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

PROOF. We are to show that if $a/b=c/d$, then

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

By Theorems E and F, we have

$$\frac{a+b}{b}=\frac{c+d}{d}, \text{ and } \frac{a-b}{b}=\frac{c-d}{d}.$$

By dividing the first of these equations by the second, member by member, we obtain the desired result, namely

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

For example, $\frac{3}{2}=\frac{9}{6}$ gives *by composition and division* the new proportion $(3+2)/(3-2)=(9+6)/(9-6)$, or $\frac{5}{1}=\frac{15}{3}$.

108. Several Equal Ratios. THEOREM H. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

PROOF. We are to show that if $a/b=c/d=e/f=g/h=\cdots$, then

$$\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\cdots.$$

Let k be the value of any one of the equal ratios, so that

$$k=\frac{a}{b},\ k=\frac{c}{d},\ k=\frac{e}{f},\ k=\frac{g}{h},\ \cdots.$$

Then $a=kb,\ c=kd,\ e=kf,\ g=kh,\ \cdots.$

Hence

$$\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=\frac{k(b+d+f+h+\cdots)}{b+d+f+h+\cdots}=k,$$

or

$$\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\cdots.$$

For example, the three equal ratios $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}$ give the new proportions

$$\frac{2+4+6}{3+6+9}=\frac{2}{3}=\frac{4}{6}=\frac{6}{9},$$

or

$$\frac{12}{18}=\frac{2}{3}=\frac{4}{6}=\frac{6}{9}.$$

EXERCISES

1. Given the proportion $\frac{5}{3}=\frac{15}{9}$. Write down the various proportions to be obtained from this by (1) inversion, (2) alternation, (3) composition, (4) division, (5) composition and division. Note that each new proportion thus obtained is a *true* one.

2. Show that if $a/b=c/d$, then $a^2/b^2=c^2/d^2$; in other words, if four numbers are in proportion, their squares are also in proportion.

3. Show that if four numbers are in proportion, their cubes are also in proportion; likewise their square roots; likewise their cube roots.

4. Show by means of Theorem A (§ 98) that $a/b=a^2/b^2$ is *not* a true proportion, unless $a=b$. In other words, the ratio of two numbers is not in general the same as the ratio of their squares. Prove similarly that the ratio of two numbers is not in general the same as the ratio of their cubes, or of their square roots, or of their cube roots.

5. If $a:b=c:d$, establish the following proportions.

(*a*) $a^2:b^2c^2=1:d^2$.

SOLUTION. It suffices to show here that the product of the extremes is equal to the product of the means, for if these two products are the same, the proportion in question is true by Theorem B. Thus we are to prove that $a^2d^2=b^2c^2$.

Now, we know (by hypothesis) that $a:b=c:d$, or $ad=bc$. Squaring gives, as desired, $a^2d^2=b^2c^2$, thus completing the proof.

(*b*) $ac:bd=c^2:d^2$.

[HINT. Remember to use the hypothesis, namely that $a:b=c:d$.]

(*c*) $\sqrt{ad}:\sqrt{b}=\sqrt{c}:1$. (*d*) $a:a+b=a+c:a+b+c+d$.

(*e*) $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

(*f*) $a+b+c+d:a-b+c-d=a+b-c-d:a-b-c+d$.

(*g*) If $a:b=c:d$, and $x:y=z:w$, show that $ax:cz=by:dw$.

CHAPTER XVII

VARIATION

109. Direct Variation. One quantity is said to ***vary directly as*** another when the two are so related that, though the quantities themselves may change, their *ratio* never changes.

Thus the amount of work a man does varies directly as the number of hours he works. For example, if it takes him 4 hours to draw 10 loads of sand, we can say it will take him 8 hours to draw 20 loads. Here the first ratio is $\frac{4}{10}$ and the second is $\frac{8}{20}$ and the two are seen to be equal, though the numbers in the second have been changed from what they were in the first. In general, if the man works twice as long, he will draw twice as much; if he works three times as long, he will draw three times as much, etc.; all of which implies that the ratio of the time he works to the amount he draws in that time never changes.

EXERCISES

Determine which of the following statements are true and which false, giving your reason in each instance.

1. The amount of electricity used in lighting a room varies directly as the number of lights turned on.

2. The amount of water in a cylindrical pail varies directly as the height to which the water stands in the pail.

3. The amount of gasoline used by an automobile in any given time (one week, say) varies directly as the amount of driving done.

4. The time it takes to walk from one place to another at any given rate (3 miles an hour, say) varies directly as the distance between the two places.

5. The time it takes to walk any given distance (5 miles, say) varies directly as the rate of walking.

6. The perimeter of a square varies directly as the length of one side.

7. The circumference of a circle varies directly as the length of the radius.

8. The area of a square varies directly as the length of one side.

9. x varies directly as $10\,x$.

10. x varies directly as $10\,x^2$.

110. Inverse Variation One quantity, or number, is said to ***vary inversely as*** another when the two are so related that, though the quantities themselves may change, their *product* never changes.

Thus the time occupied in doing any given piece of work varies inversely as the number of men employed to do it. For example, if it takes 2 men 6 days, it will take 4 men only 3 days. The point to be observed here is that the first product, 2×6, equals the second product, 4×3. In general, if twice as many men are employed it will take *half* as long; if three times as many men are employed, it will take *one third* as long, etc. In all these cases, the number of men employed multiplied by the corresponding time required to do the work remains the same.

NOTE. The term *varies inversely as* is due to the fact that in case xy never changes (as required by the above definition), it follows that $x\div(1/y)$ never changes, since $xy=x\div(1/y)$. That is, x varies directly as the reciprocal, or *inverse*, of y (§ 109).

EXERCISES

Determine which of the following statements are true and which false, giving your reason in each instance.

1. The time it takes water to drain off a roof varies inversely as the number of (equal sized) conductor pipes.

2. The time it takes to walk any given distance (5 miles, say) varies inversely as the rate of walking. (Compare Ex. 5, p. 179.)

3. The weight of a pail of water varies inversely as the amount of water that has been poured out of it.

4. x varies inversely as $10/x$.

5. x varies inversely as $10/x^2$.

111. Joint Variation. One quantity, or number, is said to ***vary jointly as*** two others when it varies directly as their product.

Thus the area of a triangle varies jointly as its base and altitude, for if A be the area of any triangle and b its base and h its altitude, we have $A = \frac{1}{2}\,bh$, which may be written $A/bh = \frac{1}{2}$. Whence A varies directly as the product bh (§ 109), that is the ratio of A to bh is always the same, namely $\frac{1}{2}$ in this instance.

EXERCISES

Determine whether the following statements are true, giving your reason in each instance.

1. The area of a rectangle varies jointly as its two dimensions, that is as its length and breadth.

2. The pay received by a workman varies jointly as his daily wage and the number of days he works.

3. The amount of reading matter in a book varies jointly as the thickness of the book and the distance between the lines of print on the page.

4. The interest received in one year from an investment varies jointly as the principal and rate.

5. The volume of a rectangular parallelepiped (such as an ordinary rectangular shaped box) varies jointly as its length, breadth, and height.

[HINT. Here we have one quantity varying jointly as *three* others. First make a definition yourself of what such variation means.]

112. Variables and Constants. When we say that the amount of work a man does varies directly as the number of hours he works (see § 109), we are dealing with two quantities, namely the amount of work done and the time used in doing it. But it is to be observed that these are not being regarded as fixed quantities, but rather as changeable ones, the only essential idea being that their *ratio* never changes. In general, quantities which are thus changeable throughout any discussion or problem are called ***variables***, while quantities which do not change are called ***constants***.

113. The Different Types of Variation Stated as Equations. We may now state very briefly and concisely what is meant by the different types of variation mentioned in §§ 109–111 and certain other important types also. To do this, let us think of x, y, and z as being certain variables and k as being some constant. Then

(1) *To say that x varies directly as y means* (by § 109) *that*

$$\frac{x}{y}=k, \quad \text{or} \quad x=ky.$$

(2) *To say that x varies inversely as y means* (by § 110) *that*

$$xy=k, \quad \text{or} \quad x=\frac{k}{y}.$$

(3) *To say that x varies jointly as y and z means* (by § 111) *that*

$$\frac{x}{yz}=k, \quad \text{or} \quad x=kyz.$$

Two other important types of variation are described below:

(4) *To say that x varies directly as the square of y means that*

$$\frac{x}{y^2}=k, \quad \text{or} \quad x=ky^2.$$

(5) *To say that x varies inversely as the square of y means that*

$$xy^2=k, \quad \text{or} \quad x=\frac{k}{y^2}$$

In all these types of variation it is important to observe that the value which must be given to the constant k depends upon the particular statement or problem in hand. For example, consider the statement that "The area of a rectangle varies jointly as its two dimensions." This means (see (3)) that if we let A be the variable area and a and b the variable dimensions, then $A=kab$. But in this case we know by arithmetic that $A=ab$, so the value of k here must be 1. On the other hand, consider the statement that "The area of a triangle varies jointly as its base and altitude." Letting A be the variable area and b and h the variable base and altitude, respectively, this means that $A=kbh$. But here, as we know from arithmetic, $k=\frac{1}{2}$.

EXERCISES

Convert each of the following statements into equations, supplying for each the proper value for the constant k mentioned in § 113.

1. The circumference of a circle varies directly as the radius.

[Hint. Let C stand for circumference and r for radius.]

2. The circumference of a circle varies directly as the diameter.

3. The area of a circle varies directly as the square of the radius.

4. The area of a circle varies directly as the square of the diameter.

5. The area of a sphere varies directly as the square of the radius. (See § 14, (*f*).)

6. The volume of a rectangular parallelepiped varies jointly as its length, breadth, and height. (See Ex. 5, p. 181.)

7. Interest varies jointly as the principal, rate, and time.

8. The volume of a sphere varies directly as the cube of the radius.

[HINT. First supply for yourself the definition of what this type of variation means.]

9. The volume of a circular cone varies jointly as the altitude and the square of the radius of the base. (See p. 103.)

10. The distance, measured in feet, through which a body falls if dropped vertically downward from a position of rest (as from a window ledge) varies directly as the square of the number of seconds it has been falling.

[HINT. It is found by experiments in physics that the value of the constant k is in this case 32 (approximately).]

11. The following, like Ex. 10, are statements of well-known physical laws. Convert each into an equation without, however, attempting to supply the proper value of k, since to do so requires a study of physics.

(*a*) If a body is tied to a string and swung round and round in a circle (as in swinging a pail of water at arm's length from the shoulder), the force, F, with which it pulls outward from the center (called *centrifugal force*) varies directly as the square of the velocity of the motion.

(*b*) The intensity of the illumination due to any small source of light (such as a candle) varies inversely as the square of the distance of the object illuminated from the source of light.

(*c*) When an elastic string is stretched out, as represented in Fig. 66, the tension (force tending to pull it apart at any point) varies directly as the length to which the string has been stretched (*Hooke's Law*).

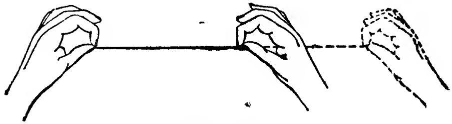


Fig. 66.

(*d*) The pressure per square inch which a given amount of gas (such as air, or hydrogen, or oxygen, or illuminating gas) exerts upon the sides of the receptacle which holds the gas (such as a bag) varies inversely as the volume of the receptacle (*Boyle's Law*).

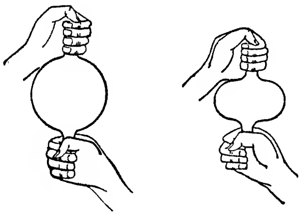


Fig. 67.

For example, whenever air is confined in a rubber balloon, as in the first drawing in Fig. 67, it exerts a certain pressure upon each square inch of the interior surface. If the balloon be squeezed, as in the second drawing in Fig. 67 (no air being allowed to escape), until its volume is half of what it was before, this pressure will be exactly doubled.

114. Problems in Variation. The problems naturally arising in the study of variation fall into two general classes as follows:

(1) Those in which the value of the constant k mentioned in § 113 can be determined from the statement of the problem and forms an essential part in the solution. This kind of problem is illustrated by Exs. 1–10 on pages 185–187. The solution given for Ex. 1 should be well understood before the pupil undertakes Exs. 2–10.

(2) Those in which it is *not* necessary to know the value of k. Such problems are illustrated in Exs. 11–20, pp. 187–189.

The pupil is advised to work several problems from each group rather than to confine his attention to either.

EXERCISES

I. Illustrations of Case (1)

1. In a fleet of ships all made from the same model (that is, of the same shape, but of different sizes) the area of the deck varies directly as the square of the length of the ship. If the ship whose length is 200 feet has 5000 square feet of deck, how many square feet in the deck of the ship which is 300 feet long?

Solution. Let A represent the area of deck on the ship whose length is l. Then the given law of variation, expressed as an equation (§ 113), is

$$(1) \qquad A = kl^2. \quad (k = \text{some constant})$$

Since the ship which is 200 feet long has 5000 square feet of deck, it follows from (1) that we must have

$$5000 = k(200)^2.$$

This equation tells us that the value of k in the present problem must be

$$k = \frac{5000}{(200)^2} = \frac{5000}{200 \times 200} = \frac{1}{8}.$$

Placing this value of k in (1), gives us an equation which determines *completely* the relation between A and l in the present problem, that is

$$(2) \qquad A = \tfrac{1}{8} l^2.$$

Now the problem asks how many square feet of deck there are in the ship whose length is 300 feet. This can be found by simply placing $l = 300$ in (2) and solving for A. Thus

$$A = \frac{1}{8} \times (300)^2 = \frac{300 \times 300}{8} = 11{,}250 \text{ square feet.} \quad \textit{Ans.}$$

NOTE. Observe that the first step in the above solution is to express *as an equation* the law of variation belonging to the problem. Next, the constant k is determined. After this, the first equation is rewritten in its more exact form obtained by assigning to k its value. The answer is then readily obtained.

These steps should be followed in working each of the Exs. 2–10.

2. In a fleet of ships all of the same model, the ship whose length is 200 feet contains 6000 square feet in its deck. How long must a similar ship be made if its deck is to contain 13,500 square feet?

3. To make a suit of clothes for a man who is 5 feet 8 inches high requires 6 square yards of cloth. How much cloth will be required to make a suit for a man of similar build, whose height is 6 feet 2 inches?

[HINT. The areas of any two similar figures vary directly as the *squares* of their heights.]

4. If 10 men can do a piece of work in 20 days, how long will it take 25 men to do it?

[HINT. The time required varies inversely as the number of men employed.]

5. The horsepower required to propel a ship varies directly as the cube of the speed. If the horsepower is 2000 at a speed of 10 knots, what will it be at a speed of 15 knots?

6. A silver loving-cup (such as is sometimes given as a prize in athletic contests) is to be made, and a model is first prepared out of wood. The model is 8 inches high and weighs 12 ounces. What will the loving-cup cost if made 10 inches high, it being given that silver is 17 times as heavy as wood and costs \$2.20 an ounce?

[HINT. The volumes and hence the weights of any two similar figures vary directly as the *cubes* of their heights. See § 99 (*c*).]

7. When electricity flows through a wire, the wire offers a certain resistance to its passage. The unit of this resistance is called the *ohm*, and for a given length of wire the resistance varies inversely as the square of the diameter. If a certain length of wire whose diameter is $\frac{1}{4}$ inch offers a resistance of 3 ohms, what will be the resistance of a similar wire (same length and material) $\frac{1}{3}$ of an inch in diameter?

8. Three spheres of lead whose radii are 6 inches, 8 inches, and 10 inches respectively are melted and made into one. What is the radius of the resulting sphere?

9. On board a ship at sea the distance of the horizon varies directly as the square root of one's height above the water. If, at a height of 20 feet, the horizon is 5.5 miles distant, what is its distance as seen from a light-house 80 feet above sea-level?

10. The horsepower that a shaft can safely transmit varies jointly as its speed in revolutions per minute and the cube of its diameter. A 3-inch steel shaft making 100 revolutions per minute can transmit 85 horsepower. How many horsepower can a 4-inch shaft transmit at a speed of 150 revolutions per minute?

II. Illustrations of Case (2)

11. Knowing that the force of gravitation due to the earth varies inversely as the square of the distance from the earth's center (Newton's Law of Gravitation), find how far above the earth's surface a body must be taken in order to lose half its weight.

Solution. Letting W represent the weight of a given body at the distance d from the earth's center, the law stated above, when expressed as an equation, becomes

(1) $$W = \frac{k}{d^2}. \quad (k = \text{some constant})$$

Now let W_1 represent the weight of the body when on the surface. Remembering that the earth's radius is 4000 miles (approximately), equation (1) gives

(2) $$W_1 = \frac{k}{4000^2}.$$

Next, let x represent the desired distance, namely the distance above the surface at which the same body loses half its weight. At this distance its weight will consequently be $\frac{1}{2}W_1$, while its distance from the earth's *center* is now $4000+x$. So (1) gives

(3) $$\frac{W_1}{2} = \frac{k}{(4000+x)^2}.$$

Dividing equation (3) by equation (2), noting the cancelation of W_1 on the left and of the (unknown) k on the right, we obtain

$$\frac{1}{2} = \frac{4000^2}{(4000+x)^2}.$$

It remains only to solve this equation for x.

Clearing of fractions, $(4000+x)^2 = 2 \cdot 4000^2 = 4000^2 \cdot 2$.

Extracting the square root of both members, $4000+x = 4000\sqrt{2}$.

Solving, $x = 4000\sqrt{2} - 4000 = 4000(\sqrt{2}-1)$ miles. *Ans.*

To find the approximate value of this answer, we have (see table)

$$\sqrt{2} = 1.41421$$

so that $x = 4000(1.41421-1) = 4000 \times .41421 = 1656.84$ miles. *Ans.*

NOTE. Observe that the first step in the above solution (as also in the preceding exercises) is to express *as an equation* the law of variation belonging to the problem. Then write down the two special equations which express the particular conditions given in the problem and divide one of these equations by the other to eliminate the unknown k. The answer is then readily obtained. A similar process should be followed in working the remaining exercises of this list.

12. Show that the earth's attraction at a point on the surface is over 5000 times as strong as at the distance of the moon, that is at the (approximate) distance of 280,000 miles.

[HINT. Call W_1 the weight of a given body on the surface, and let W_2 represent the weight of the same body at the distance of the moon from the earth's center. Then use the law expressed in (1) of the solution of Ex. 11.]

13. A book is being held at a distance of 2 feet from an incandescent lamp. How much nearer must it be brought in order that the illumination on the page shall be doubled? (See Ex. 11 (*b*), p. 183.)

14. If two like coins (such as quarter dollars) were melted and made into a single coin of the same thickness as the original, show that its diameter would be $\sqrt{2}$ times as great.

[HINT. Call D the diameter of the given coins and A the area of each. Note that the area of the new coin will then be $2A$. Use the result stated in Ex. 3, p. 182.]

15. Find the result in Ex. 14 when four equal-sized coins are used.

16. Show that a falling body will pass over the second 3 feet of its descent in about .4 of the time it takes it to pass over the first 3 feet. (See Ex. 10, p. 183.)

17. The time required for a pendulum to make a complete oscillation (swing forward and back) varies directly as the square root of its length. By how much must a 2-foot pendulum be shortened in order that its time of complete oscillation may be halved?

18. If the diameter of a sphere be increased by 10%, by what per cent will the volume be increased?

19. Show that if a city is receiving its water supply by means of a main (large pipe) from a reservoir, the supply can be increased 25% by increasing the diameter of the main by about 12%.

20. It is desired to build a ship similar in shape to one already in use but having a 40% greater cargo space (or hold). By what per cent must the beam (width of the ship) be increased. (See § 99 (*c*).)

115. Variation Geometrically Considered. If a variable y varies directly as another variable x, we know (§ 113) that this is equivalent to having the equation $y=kx$, where k is some constant. If the value of k is 1, this equation takes the definite form $y=x$, and we may now draw its graph, the result being a certain straight line. If, on the other hand, $k=2$, we have $y=2x$, and this again is an equation whose

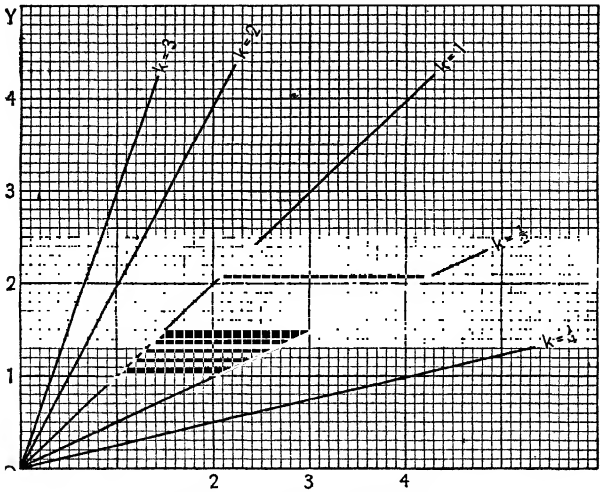


Fig. 68. — Direct Variation.

graph may be drawn, leading to a straight line, but a different one. In general, whatever the value of k, the corresponding equation has a straight-line graph. The fact that in all cases the graph is a *straight line* characterizes this type of variation, that is, characterizes the type in which one variable varies *directly* as another. The figure shows the lines corresponding to several different values of k.

In case a variable y varies inversely as another variable x, we know (§ 113) that there exists an equation of the form $y=k/x$, where k is some constant. If we let $k=1$, this becomes $y=1/x$. By letting x take a series of values and determining the corresponding values of y from this equation (thus forming a table as in § 57) we obtain the graph. Similarly, corresponding to the value $k=2$ we have $y=2/x$,

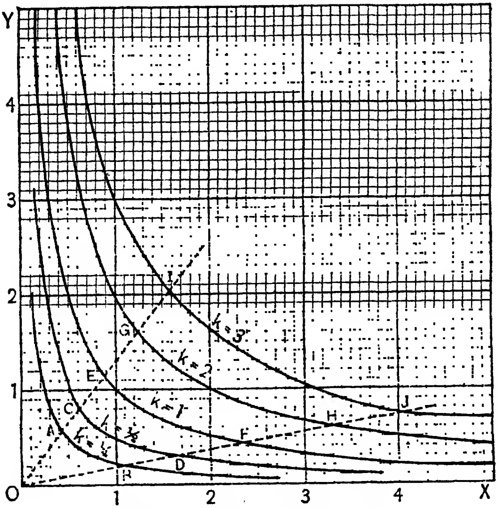


FIG. 69.—INVERSE VARIATION.

and this equation has a definite graph which is different from the one just mentioned. In general, whatever the value of k, the corresponding equation has a graph, but it is now to be noted that these graphs are *not* straight lines; they are *hyperbolas*. (See Ex. 2, § 78.) The figure shows the curves corresponding to several different values of k.

NOTE. Though these curves differ in form, they have the following feature in common: Through the origin draw any two straight

lines (dotted in figure). Then the intercepted arcs AB, CD, EF, GH, etc., are similar, that is the smallest arc when simply *magnified* by the proper amount produces one of the others.

EXERCISES

Draw diagrams to represent the geometric meaning of each of the following statements.

1. y varies directly as the square of x.

2. y varies inversely as the square of x.

3. y varies as the cube of x.

4. y varies directly as x, and $y=6$ when $x=2$.

[HINT. The diagram here consists of a single line.]

5. y varies inversely as x, and $y=6$ when $x=2$.

6. The cost of n pounds of butter at 40¢ per pound is $c=40\,n$.

7. The amount of the extension, e, of a stretched string is proportional to the tension, t, and $e=2$ in. when $t=10$ lb. (See Ex. 11 (c), p. 184.)

8. The pressure, p, of a gas on the walls of a retaining vessel varies inversely as the volume, v; and $p=40$ lb. per square foot when $v=10$ cu. ft.

9. The length, L, of any object in centimeters is proportional to its length, l, expressed in inches; and $L=2.54$ when $l=1$.

NEWTON

(*Sir Isaac Newton,* 1642–1727)

Discoverer of the law of gravitation and famous in algebra for his discovery of the binomial theorem. Inventor of the branch of higher mathematics called the Calculus, wherein rates of motion and other changing, or variable, quantities are extensively studied.

CHAPTER XVIII

EXPONENTS

I. Positive Integral Exponents

116. Powers. Involution. Just as $a^2 = a \cdot a$; $a^3 = a \cdot a \cdot a$; etc., so we define the nth power of a, where n is any positive integer, as follows:

$$a^n = a \cdot a \cdot a \cdot a \cdot a \cdots a \text{ (} n \text{ factors).}$$

The process of finding the power of a number, or expression, is called ***involution.***

117. Laws of Exponents. There are five fundamental laws of exponents which are as follows, it being understood that m and n everywhere stand for positive integers:

I. Multiplication Law. This law for multiplying two powers of the same quantity is

$$\boldsymbol{a^m \cdot a^n = a^{m+n}}.$$

Proof.

$$a^m = a \cdot a \cdot a \cdot a \cdot a \cdots a \text{ (} m \text{ factors).} \qquad (\S\ 116)$$
$$a^n = a \cdot a \cdot a \cdot a \cdot a \cdots a \text{ (} n \text{ factors).}$$

Hence

$$a^m \cdot a^n = \{a \cdot a \cdot a \cdot a \cdots a \text{ (} m \text{ factors)}\} \cdot \{a \cdot a \cdot a \cdot a \cdots a \text{ (} n \text{ factors)}\}$$
$$= a \cdot a \cdot a \cdot a \cdots a \text{ (} m+n \text{) factors} = a^{m+n}. \qquad (\S\ 116)$$

Therefore

$$a^m \cdot a^n = a^{m+n}.$$

Illustrations.

$$2^2 \cdot 2^3 = 2^{2+3} = 2^5; \quad (-3)^3 \cdot (-3)^7 = (-3)^{10};$$
$$x^8 \cdot x^{15} = x^{23}; \quad (a+b)^2 \cdot (a+b)^4 = (a+b)^6.$$

II. Division Law. The law for dividing one power by another power of the same quantity is

$$\boldsymbol{a^m \div a^n = a^{m-n}.}$$

Proof. $a^m \div a^n = \dfrac{a^m}{a^n} = \dfrac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdots a \cdot a \cdot a \cdots a \text{ (} m \text{ factors)}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdots \cancel{a} \text{ (} n \text{ factors)}}$

$= a \cdot a \cdot a \cdot a \cdots a$ $(m-n)$ factors $= a^{m-n}$. (§ 116)

Therefore

$$a^m \div a^n = a^{m-n}.$$

Illustrations.

$$3^6 \div 3^2 = 3^{6-2} = 3^4; \quad (-2)^5 \div (-2)^3 = (-2)^2;$$
$$x^8 \div x^5 = x^3; \quad (a+b)^7 \div (a+b)^3 = (a+b)^4.$$

III. Law for the Power of a Power. The law for raising a power of a quantity to a new power is

$$\boldsymbol{(a^m)^n = a^{mn}.}$$

Proof. $(a^m)^n = a^m \cdot a^m \cdot a^m \cdot a^m \cdots a^m$ (n factors) (§ 116)

$= a^{m+m+m+m+\cdots+m \text{ (} n \text{ terms)}}$. (Law I)

Therefore $(a^m)^n = a^{mn}$ since $m + m + m + \cdots + m$ to n terms $= mn$.

Illustrations.

$$(4^2)^3 = 4^{2\times 3} = 4^6; \quad \{(-2)^3\}^3 = (-2)^9;$$
$$(x^4)^5 = x^{20}; \quad \{(a+b)^3\}^4 = (a+b)^{12}.$$

IV. Law for the Power of a Product. The law for raising to a power a product of two quantities is

$$\boldsymbol{(ab)^n = a^n b^n.}$$

Proof.

$(ab)^n = (ab) \cdot (ab) \cdot (ab) \cdots (ab)$ (n factors) (§ 116)

$= \{a \cdot a \cdot a \cdots a \text{ (} n \text{ factors)}\} \cdot \{b \cdot b \cdot b \cdot b \cdots b \text{ (} n \text{ factors)}\}$

$= a^n b^n$.

Therefore

$$(ab)^n = a^n b^n.$$

Illustrations.

$$(2\times 3)^4 = 2^4 \times 3^4; \quad \{(-3)(-2)\}^3 = (-3)^3(-2)^3;$$
$$(xy)^5 = x^5 y^5; \quad \{(a+b)(c+d)\}^3 = (a+b)^3(c+d)^3.$$

V. Law for the Power of a Quotient. The law for raising to a power the quotient of two quantities is

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Proof. $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdots \left(\frac{a}{b}\right)$ (n factors) (§ 116)

$$= \frac{a \cdot a \cdot a \cdots a \ (n \text{ factors})}{b \cdot b \cdot b \cdots b \ (n \text{ factors})} = \frac{a^n}{b^n}.$$

Therefore $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$

Illustrations. $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$; $\left(\frac{-4}{3}\right)^2 = \frac{(-4)^2}{3^2}$;

$\left(\frac{x}{y}\right)^7 = \frac{x^7}{y^7}$; $\left(\frac{a+b}{a-b}\right)^5 = \frac{(a+b)^5}{(a-b)^5}$.

EXERCISES

Find the results of the indicated operations in the following cases, using one (or more) of the five laws in § 117.

1. $2^5 \cdot 2^3$.
2. $(-1)^3 \cdot (-1)^2$.
3. $(\frac{2}{3})^2 \cdot (\frac{2}{3})^4$.
4. $x^{10} \cdot x^2$.
5. $m^{12} \cdot m^{13}$.
6. $y^5 \cdot y^n$.
7. $q^m \cdot q^4$.
8. $t^{2a} \cdot t^a$.
9. $z^{r-1} \cdot z^{r+1}$.
10. $w^{m-3} \cdot w^{m+4}$.
11. $g^{p+q-1} \cdot g^{1+r}$.
12. $8^3 \div 8^2$.
13. $(-3)^5 \div (-3)^3$.
14. $(\frac{4}{5})^9 \div (\frac{4}{5})^7$.
15. $x^{10} \div x^2$.
16. $m^{12} \div m^6$.
17. $y^5 \div y^n$.
18. $q^m \div q^4$.
19. $t^{2a} \div t^a$.
20. $z^{r+1} \div z^{r-1}$.
21. $w^{m+4} \div w^{m-3}$.
22. $g^{w+p-1} \div g^{1+r}$.
23. $(a+b)^{2r} \div (a+b)^{r-1}$.
24. $(2^5)^3$.
25. $\{(-8)^3\}^4$.
26. $(x^6)^4$.
27. $(y^3)^7$.
28. $(m^4)^8$.
29. $(a^2b)^3$.

[Hint to Ex. 29. First use Law IV, then Law III.]

30. $(x^3y^2)^2$.
31. $(abc)^3$.
32. $(a^2b^3c)^4$.
33. $(m^2n^3w^4)^3$.
34. $\{(a+b)^2(c+d)^3\}^4$.
35. $(x^4y^3)^{2m}$.
36. $(r^2s^6t^3)^n$.
37. $\left(\frac{3}{5}\right)^3$.
38. $\left(\frac{m^5}{n}\right)^4$.
39. $\left(\frac{r^n}{s^m}\right)^2$.

40. $\left(\frac{x^a}{y}\right)^3$.

41. $\left(\frac{x^2-y}{x^2+y}\right)^3$.

42. $\left(\frac{x^2}{y}\right)^3 \cdot \left(\frac{x}{y^2}\right)^3$.

43. $\left(\frac{x^2}{y}\right)^3 \div \left(\frac{x}{y^2}\right)^3$.

44. $\left(-\frac{m^r}{n^s}\right)^t$.

45. $\left(-\frac{x^{2n}}{y^{3m}}\right)^k$.

118. Roots. Evolution. Just as $\sqrt{a}$ means the number whose *square* gives a, and $\sqrt[3]{a}$ means the number whose *cube* gives a, etc., so we define ***the nth root of a***, $\sqrt[n]{a}$, to be the number whose nth *power* gives a, that is we agree that

$$(\sqrt[n]{a})^n = a.$$

Thus $\sqrt[3]{x^{12}} = x^4$, because $(x^4)^3 = x^{12}$. (§ 117, Law III.) Similarly $\sqrt[4]{a^{16}b^8} = a^4b^2$, because $(a^4b^2)^4 = a^{16}b^8$.

NOTE. In case $n = 2$, we write simply $\sqrt{\ }$ instead of $\sqrt[2]{\ }$.

The number n is called the ***index*** of the root.

The number under the sign $\sqrt{\ }$, as a, is called the ***radicand***.

The process of finding the root of a number or expression is called ***evolution***.

119. Rule for Finding the nth Root of an Expression. The nth root of an expression may be obtained readily in case the expression itself is an exact nth power. This is illustrated in the following examples.

EXAMPLE 1. To find the value of $\sqrt[3]{m^6n^9}$.

SOLUTION. The expression m^6n^9 may be written as an exact cube, namely $(m^2n^3)^3$. (Laws IV and III of § 117.)

Therefore $\sqrt[3]{m^6n^9} = \sqrt[3]{(m^2n^3)^3} = m^2n^3$. *Ans.* (§ 118)

EXAMPLE 2. To find the value of $\sqrt[5]{\frac{x^{10}y^5}{z^{15}}}$.

SOLUTION. The expression $\frac{x^{10}y^5}{z^{15}}$ may be written as an exact 5th power, namely $\left(\frac{x^2y}{z^3}\right)^5$. (Laws IV and III of § 117.)

Therefore $\sqrt[5]{\frac{x^{10}y^5}{z^{15}}} = \sqrt[5]{\left(\frac{x^2y}{z^3}\right)^5} = \frac{x^2y}{z^3}$. *Ans.* (§ 118)

Observe that the answer to Example 1 (namely m^2n^3) is the result of simply dividing each exponent of the radicand (namely m^6n^9) by the index of the desired root (namely 3). Similarly, in Ex. 2 if we simply divide each exponent in $\frac{x^{10}y^5}{z^{15}}$ by 5 we get the answer immediately. Thus, in practice, we use the following rule.

To find the nth root of an exact nth power, divide the exponent of each factor of the radicand by n.

Thus

$$\sqrt[4]{r^8s^{12}} = r^2s^3;\quad \sqrt[3]{\frac{x^6y^{12}}{z^9}} = \frac{x^2y^4}{z^3};\quad \sqrt{\frac{(a+b)^4(c-d)^2}{x^8y^6}} = \frac{(a+b)^2(c-d)}{x^4y^3}.$$

NOTE. It will be recalled (§ 39) that, unless otherwise stated, the symbol $\sqrt{a}$ means the *positive* number whose square is a. Thus $\sqrt{9} = +3$, the other root, -3, being represented by $-\sqrt{9}$. This agreement is made in order to bring about perfect definiteness in the use of the symbol $\sqrt{\ }$.

EXERCISES

Determine (from the definition in § 118) the value of:

1. $\sqrt[3]{8}$.
2. $\sqrt[3]{-27}$.
3. $-\sqrt[4]{81}$.
4. $\sqrt[5]{32a^5}$.
5. $\sqrt{625}$.
6. $\sqrt{\frac{9}{16}}$.
7. $\sqrt[3]{\frac{27a^6}{64b^{12}}}$.
8. $\sqrt[4]{m^{20}n^{16}q^{24}}$.

Determine by means of the Rule in § 119 the value of:

9. $\sqrt[3]{m^6n^9}$.
10. $\sqrt[6]{64a^6b^6}$. [HINT. Write 2^6 for 64.]
11. $\sqrt[4]{625a^8b^4}$.
12. $\sqrt[3]{-27m^9n^6}$.
13. $-\sqrt[4]{81x^8y^{16}}$.
14. $-\sqrt[5]{-32x^{10}y^{15}}$.
15. $\sqrt{\frac{x^6}{y^4}}$.
16. $\sqrt[4]{\frac{m^4}{81}}$.
17. $\sqrt[5]{\frac{32a^5}{x^{10}}}$.
18. $\sqrt[6]{\frac{x^6y^6z^{12}}{r^{24}s^{12}}}$.
19. $-\sqrt{a^{2m}}$.
20. $\sqrt{(-a)^{2m}}$.
21. $-\sqrt[3]{a^{3m}}$.
22. $\sqrt[3]{(-a)^{3m}}$.
23. $\sqrt[n]{x^{2n}y^{3n}}$.
24. $\sqrt[4]{\frac{81x^{4m}}{y^{16n}}}$.
25. $\sqrt[p]{\frac{x^{pq}}{y^{pr}}}$.
26. $\sqrt[3]{8(a+b)^6}$.
27. $\sqrt{(a^2+b^2)^{4n}}$.

II. Fractional, Zero, and Negative Exponents

120. Introduction of General Exponents. Thus far we have considered only positive integral exponents. Such symbols as $a^{3/4}$ and a^{-2} thus have no meaning for us as yet since there can be no such thing as taking a as a factor *three fourths* times, or *minus two times.* However, we shall now see that by extending our definitions we can assign perfectly definite meanings to these symbols as well as to all others wherein fractional, zero, or negative exponents occur.

121. Meaning of a Fractional Exponent. If $a^{3/4}$ is to obey the multiplication law (§ 117) then

$$a^{3/4}\cdot a^{3/4}\cdot a^{3/4}\cdot a^{3/4}=a^{3/4+3/4+3/4+3/4}=a^{12/4}=a^3.$$

That is

$$(a^{3/4})^4=a^3,$$

so that we must have

$$a^{3/4}=\sqrt[4]{a^3}.$$

Thus we naturally take $\sqrt[4]{a^3}$ to be the meaning of $a^{3/4}$.

Similarly (if the multiplication law is to hold true), the meaning of $a^{2/3}$ is $\sqrt[3]{a^2}$, while that of $a^{4/5}$ is $\sqrt[5]{a^4}$, etc.

So, in all cases $a^{m/n}$ means *the nth root of* a^m, that is,

$$a^{m/n}=\sqrt[n]{a^m}.$$

Thus

$$8^{2/3}=\sqrt[3]{8^2}=\sqrt[3]{64}=4.$$

Similarly,

$$(x^8y^4)^{3/4}=\sqrt[4]{(x^8y^4)^3}=\sqrt[4]{x^{24}y^{12}}=x^6y^3.\quad \text{(See Rule in § 119.)}$$

EXERCISES

Express with radical sign and find the value of:

1. $8^{1/3}$.
2. $4^{1/2}$.
3. $9^{1/2}$.
4. $27^{1/3}$.
5. $(-8)^{1/3}$.
6. $27^{2/3}$.
7. $32^{2/5}$.
8. $81^{3/4}$.
9. $64^{1/6}$.
10. $(x^6)^{1/3}$.
11. $(y^{10})^{1/5}$.
12. $(y^{10})^{2/5}$.
13. $(x^3y^6)^{2/3}$.
14. $(16\,x^8y^{16}z^4)^{1/2}$.
15. $\{(a+b)^3\}^{2/3}$.

Express with radical signs:

16. $2^{2/3}$.	**18.** $4^{3/2}$.	**20.** $(a^2)^{4/3}$.	**22.** $2\,x^{1/3}$.	**24.** $m^{2/3}n^{3/4}$.
17. $3^{2/3}$.	**19.** $a^{4/3}$.	**21.** $(3\,x)^{3/4}$.	**23.** $(2\,ab)^{3/4}$.	**25.** $(x+y)^{5/6}$.

Express with fractional exponents:

26. $\sqrt[3]{a^4}$.	**29.** $2\sqrt[4]{x^5}$.	**32.** $3\sqrt[3]{n^2}$.	**35.** $\sqrt{(a+b)^3}$.
27. $\sqrt[6]{x^5}$.	**30.** $\sqrt[4]{(-a)^2}$.	**33.** $\sqrt[3]{(3\,n)^2}$.	**36.** $\sqrt[3]{(a+b)}$.
28. $\sqrt[3]{ab}$.	**31.** $\sqrt{abc}$.	**34.** $a^2\sqrt[3]{a^5}$.	**37.** $\sqrt[4]{a^2(m-n)^3}$.

122. Meaning of a Zero Exponent. If a^0 is to obey the multiplication law (§ 117), then $a^m \cdot a^0 = a^{m+0}$, that is $a^m \cdot a^0 = a^m$. Dividing both members of the last equality by a^m gives $a^0 = a^m \div a^m = 1$. That is,

$$\mathbf{a^0 = 1.}$$

This means that *the zero power of any number a* (*except* 0) *must always be taken equal to* 1.

Thus $3^0 = 1$; $(-32)^0 = 1$; $(\frac{1}{4})^0 = 1$; $x^0 = 1$; $(mn)^0 = 1$; $(a+b)^0 = 1$; etc.

123. Meaning of Negative Exponents. If a^{-m} is to obey the multiplication law (§ 117), then $a^m \cdot a^{-m} = a^{m-n} = a^0$, that is $a^m \cdot a^{-m} = 1$ (§ 122). Dividing both members of the last equation by a^m gives

$$\mathbf{a^{-m} = \frac{1}{a^m}}.$$

This means that *a negative power of any number a must always be taken equal to* 1 *divided by the corresponding positive power of a.*

Thus $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$; $(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$; $x^{-2/3} = \frac{1}{x^{2/3}} = \frac{1}{\sqrt[3]{x^2}}$.

Similarly, $(a+b)^{-1/2} = \frac{1}{(a+b)^{1/2}} = \frac{1}{\sqrt{a+b}}$.

EXERCISES

Express with positive exponents and find the values of each of the following expressions.

1. 3^{-2}.	5. $2^{-1}\cdot 3^{-2}$.	9. $8^2\cdot 4^{-3}$.	13. $81^{-1/4}$.
2. 4^{-2}.	6. $4^0\cdot 3^{-3}$.	10. $8^{-1/3}$.	14. $64^{-1/6}$.
3. 2^{-4}.	7. $7\cdot 4^{-4}$.	11. $(-8)^{-1/3}$.	15. $(-125)^{-1/3}$.
4. 8^0.	8. $2^{-3}\cdot 8\cdot 4^{-3}$.	12. $27^{-1/3}$.	16. $(-32)^{-1/5}$.

Write with positive exponents each of the following expressions.

17. x^3y^{-4}.	20. $(2a)^{-3}b^3$.	23. $6^{-1}m^2n^{-3}$.
18. $x^{-2}y^2z^{-3}$.	21. $2^{-3}a^{-3}b^3$.	24. $(a^2bc)^{-2}$.
19. $2a^{-3}b^3$.	22. $(-m)^{-3}(-n)^{-2}$.	25. $\{a^2(m-n)\}^{-4}$.

124. Negative Exponents in Fractions. This is best understood from an example.

EXAMPLE. Write $\dfrac{a^{-4}b^3}{c^{-2}}$ with positive exponents only.

SOLUTION. $$\frac{a^{-4}b^3}{c^{-2}}=\frac{\frac{1}{a^4}\cdot b^3}{\frac{1}{c^2}}=\frac{b^3}{a^4}\div\frac{1}{c^2}=\frac{b^3}{a^4}\cdot\frac{c^2}{1}=\frac{b^3c^2}{a^4}.\quad Ans.\qquad (\S\ 123)$$

It is to be observed that the answer here results *directly* by transferring the factor a^{-4} to the denominator by changing the sign of its exponent, and transferring the factor c^{-2} to the numerator by likewise changing the sign of its exponent.

Thus we have the following important principle.

A factor may be transferred from either term (numerator or denominator) of a fraction to the other provided the sign of its exponent be changed.

Thus we may write $\dfrac{x^2y^{-3}z^{-4}}{w^{-2}t^3}=\dfrac{x^2w^2}{y^3z^4t^3}$.

Similarly, $\dfrac{4a^3b^{-3}c^{-2}}{d^{-4}e^5}=4a^3b^{-3}c^{-2}d^4e^{-5}$. Here we have written the fraction in a form having no denominator, that is as a product.

EXERCISES

Write each of the following expressions with positive exponents only.

1. $\dfrac{a^{-3}b}{c}$.
4. $\dfrac{3\,a^{-3}}{(2\,b)^{-2}}$.
7. $\dfrac{a^{3/4}b^{-1/2}}{c^{-2/3}d^{-1}}$.
2. $\dfrac{2\,x^{-2}}{y^{-1}}$.
5. $\dfrac{8\,m^4n^{-3}}{5\,r^{-2}}$.
8. $\dfrac{5\,x^3y^{-3/5}}{z^{-2/3}w^{-3}}$.
3. $\dfrac{x^2}{2\,y^{-1}}$.
6. $\dfrac{4\,a^{-3}b^2}{3\,c^{-2}d^{-1}}$.
9. $\dfrac{3(a+b)^{-1/2}c^{-1}}{2(c+d)e^{-4}}$.

Change each of the following expressions to the form of a product.

10. $\dfrac{x^2}{y^3}$.
12. $\dfrac{2\,a^3b^2}{c^{-1/4}}$.
14. $\dfrac{8\,a^2b^2c^2}{d^{-2}e^{-2}}$.
16. $\dfrac{6\,h^{-3}\,j^2}{5\,kl}$.
11. $\dfrac{a^2}{b^{-1}}$.
13. $\dfrac{3\,r^3s^{-2}}{m^{-3}n^{-2}}$.
15. $\dfrac{4\,a^2}{3\,b^2}$.
17. $\dfrac{3(a+b)^{1/2}}{4(c^2+d^2)^{-1/2}}$.

125. The Fundamental Laws for Any Rational Exponent. The five fundamental laws stated in § 117 were there proved true only for positive integral exponents, but it can be shown that they hold equally well for fractional, zero, or negative exponents. As the proof of this fact is long, it will be omitted from this text. The following illustrations of the *meaning* of the laws in such cases should, however, be carefully examined.

1. $a^6 \cdot a^{-3} \cdot a^{4/3} \cdot a^{2/3} = a^{6-3+4/3+2/3} = a^{6-3+2} = a^5$. (Law I)
2. $\dfrac{a^{5/6}}{a^{-3}} = a^{5/6-(-3)} = a^{5/6+3} = a^{\frac{23}{6}} = a^{3\frac{5}{6}}$. (Law II)
3. $(a^{-3/2})^{4/5} = a^{(-3/2)\,\cdot\,4/5} = a^{-6/5}$. (Law III)
4. $(a^{-3}b^2)^{-1/4} = a^{(-3)(-1/4)}\,\cdot\,b^{2(-1/4)} = a^{3/4}b^{-1/2}$. (Law IV)
5. $\left(\dfrac{a^{-3}}{b^2}\right)^{-2} = \dfrac{a^{(-3)(-2)}}{b^{2(-2)}} = \dfrac{a^6}{b^{-4}} = a^6b^4$. (Law V)

HISTORICAL NOTE. The idea of using exponents to mark the power to which a quantity is raised is due to the French mathematician and philosopher *Descartes* (1596–1650); see the picture facing p. 41), but he used only positive integral exponents, as in a^1, a^2, a^3, a^4, $\cdots$. The English mathematician *Wallis* (1616–1703) encouraged the use of fractional and negative exponents and caused them to be brought into general use.

EXERCISES

In the following exercises, assume that the laws of § 117 hold true for all rational exponents.

LAW I

Multiply

1. a^2 by a^{-1}.
2. a^3 by a^{-2}.
3. a^3 by a^{-3}.
4. a by a^{-4}.
5. a^{-2} by a^{-3}.
6. $a^{1/3}$ by $a^{2/3}$.
7. $x^{-1/2}$ by $x^{3/2}$.
8. $a^{1/2}b^{1/3}$ by $a^{1/2}b^{2/3}$.
9. $m^{1/2}n^{1/3}$ by $m^{3/2}n^{2/3}$.
10. $p^{-1/4}\,q$ by $4\,p^{5/4}$.
11. n^2 by bn^{-3}.
12. x^{m-n} by x^{m+n}.
13. $x^{(m+n)/2}$ by $x^{(m-n)/2}$.
14. $a^{1/2}+b^{1/2}$ by $a^{1/2}b^{1/2}$.
15. $a^{2/3}+a^{1/3}b^{1/3}+b^{2/3}$ by $a^{1/3}-b^{1/3}$.

[HINT. Follow the rule for multiplying one polynomial by another, as given in § 8.]

16. $m^{1/3}+m^{1/6}n^{1/6}+n^{1/3}$ by $m^{1/3}-m^{1/6}n^{1/6}+n^{1/3}$.

Carry out the following indicated operations.

17. $(a^{1/2}+b^{1/2})(a^{1/2}-b^{1/2})$.
18. $(x^{2/3}+y^{2/3})(x^{2/3}-y^{2/3})$.
19. $(a^{1/2}+b^{1/2})^2$.
20. $(A^{1/4}+5)(A^{1/4}-3)$.
21. $(m^{-1/2}-3)(m^{-1/2}-2)$.
22. $(x^{1/2}-y^{-1})^2$.
23. $(x^{2/3}-x^{1/3}y^{1/3}+y^{2/3})(x^{1/3}+y^{1/3})$.
24. $(x^{2/5}-x^{1/5}y^{-1/5}+y^{-2/5})(x^{1/5}+y^{-1/5})$.
25. $(2\,x^{2/3}-3\,x^{1/3}+4)(2+3\,x^{-1/3})$.
26. $(x^2-xy+2\,y^2)(2\,x^{-2}+x^{-1}y^{-1}+y^{-2})$.

Law II

Divide

27. a^4 by a^5. **29.** x^2 by x^{-2}. **31.** $(mn)^{2/3}$ by $(mn)^{4/3}$.

28. a^3 by a^0. **30.** $x^{3/2}$ by $x^{-1/2}$. **32.** $x^{1/2}y^{1/2}$ by $y^{-1/2}$.

33. $x^4+x^2y^2+y^4$ by x^2y^2.

Solution. $\dfrac{x^4+x^2y^2+y^4}{x^2y^2}=\dfrac{x^4}{x^2y^2}+\dfrac{x^2y^2}{x^2y^2}+\dfrac{y^4}{x^2y^2}=\dfrac{x^2}{y^2}+1+\dfrac{y^2}{x^2}$. *Ans.*

34. a^3+a^2+a by a^4.

35. $a^{-4}+a^{-2}b+b^2$ by $a^{-2}b$.

36. $x^4+2\,ax^3+5\,x^{-1}y-a^2y^{-3}+y^4$ by $x^{-1}y^{-2}$.

37. $a-b$ by $a^{1/2}+b^{1/2}$.

Solution.
$$\begin{array}{l|l}
a \qquad\qquad -b & a^{1/2}+b^{1/2} \\ \cline{2-2}
\underline{a+a^{1/2}b^{1/2}\qquad} & a^{1/2}-b^{1/2}. \quad \textit{Ans.} \\
\quad -a^{1/2}b^{1/2}-b & \\
\quad \underline{-a^{1/2}b^{1/2}-b} &
\end{array}$$

38. $a-b$ by $a^{1/2}-b^{1/2}$. **41.** $x-1$ by $x^{2/3}+x^{1/3}+1$.

39. $a+b$ by $a^{1/3}+b^{1/3}$. **42.** $x-y$ by $x^{1/4}-y^{1/4}$.

40. x^2+y^2 by $x^{2/3}+y^{2/3}$. **43.** m^2-n^3 by $m^{1/3}+n^{1/2}$.

Law III

Simplify

44. $(a^{1/2})^3$.

Solution. As in Illustration 3 of § 125, the answer is $a^{1/2\times 3}$, or $a^{3/2}$.

45. $(a^{1/2})^2$. **46.** $(x^{-1/3})^6$. **47.** $(x^{-5})^2$. **48.** $(8^{-1/3})^2$. **49.** $(16^{-1/2})^3$.

50. $\sqrt{a^{-2}}$. [Hint. $\sqrt{a^{-2}}=(a^{-2})^{1/2}$ by § 121.]

51. $\sqrt[4]{a^{-1/2}}$. **52.** $\sqrt[3]{x^{3/2}}$.

Laws IV and V

Simplify

53. $(a^{-4}b^{2/3})^{-1/2}$.

Solution. As in Illustration 4 of § 125, we have $(a^{-4}b^{2/3})^{-1/2} = a^{(-4)(-1/2)} \cdot b^{2/3(-1/2)} = a^2b^{-1/3}$. *Ans.*

54. $(a^{1/3}b^{-1/2})^6$.

55. $(x^{4/3}y^{-3})^{-1/3}$.

56. $\sqrt[3]{a^{-1/2}b^{-3}}$.

57. $\sqrt{x^{4/3}y^{-3}}$.

58. $(\frac{1}{8}\,x^9)^{1/3}$.

59. $(\frac{1}{9}\,m^{-1}n^{-1/2})^{1/2}$.

60. $\sqrt{4\,x^{-8}y^{1/4}z^{-3/2}}$.

61. $\sqrt[n]{a^xb^y}$.

62. $\left(\dfrac{x^{1/}}{y^{1/2}}\right)^2$.

[Hint to Ex. 62. See Illustration 5 in § 125.]

63. $\sqrt[3]{\dfrac{m^{-3}n^{-6}}{q^{12}}}$.

64. $\sqrt{\dfrac{x^{2n}y^{4m}}{z^{8q}w^{6r}}}$.

* MISCELLANEOUS EXERCISES

Expand, by use of Formulas VI and VII of § 10.

1. $(x^{1/2}-y^{1/2})^2$.

2. $(a^{-1}+b^{-1})^2$.

3. $(a^{1/3}+b^{1/3})^2$.

4. $(1+2\,x^{1/2})^2$.

Simplify, expressing results with positive exponents.

5. $\left(\dfrac{\sqrt{2\,x}}{\sqrt[3]{3\,y}}\right)^{-4}$.

6. $\dfrac{\sqrt[3]{a^2}\times\sqrt{b^3}}{\sqrt[4]{b^6}\times\sqrt[6]{a^{-2}}}$.

7. $\dfrac{\sqrt[3]{x^{-1}\sqrt{y^3}}}{\sqrt{y^{-1}\sqrt[3]{x^2}}}$.

8. $\sqrt[3]{mn^{-2}q^{-1}}\times\sqrt[6]{mn^4q^2}$.

9. $\left(\dfrac{x^{n+1}}{x}\right)^n\div\left(\dfrac{x}{x^{1-n}}\right)^{n-1}$.

10. $\sqrt[4]{\sqrt[3]{(16^{-2})^6}}$.

Solve for x and check each result in the following equations.

11. $x^{3/4}=8$. [Hint. Write $x^{3/4}$ in the form $(x^{1/4})^3$.]

12. $x^{2/5}=9$.

13. $x^{4/3}=16$.

14. $\frac{1}{3}\,x^{3/2}=72$.

15. $\frac{1}{4}\,x^{2/3}=25$.

16. $25\,x^{-2/3}=1$.

17. $x^{-3/2}-27=0$.

CHAPTER XIX

RADICALS

126. Important Formulas. In § 3 we defined $\sqrt{a}$ as meaning that number or expression which, when raised to the nth power, would give a; that is

(1) $$(\sqrt[n]{a})^n = a.$$

Unless a is an exact nth power of some number or expression, we agreed (§ 41) to call $\sqrt[n]{a}$ a *radical of the nth order.*

Thus $\sqrt{5}$, $\sqrt{23}$, $\sqrt{\frac{2}{3}}$, $\sqrt{.05}$, $\sqrt{x+y}$, $\sqrt{m^2+n^2}$ are radicals of the second order; $\sqrt[3]{5}$, $\sqrt[3]{7}$, $\sqrt[3]{\frac{3}{4}}$, $\sqrt[3]{x+y}$ are radicals of the third order, etc.

Moreover, we saw (§ 44) that there exist two general formulas as follows:

(2) $$\sqrt{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b},$$

(3) $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

And, in connection with the study of fractional exponents, we have seen (§ 121) that the meaning of $a^{1/n}$ must be taken to be $\sqrt{a}$, that is we have the formula

(4) $$a^{1/n} = \sqrt{a}.$$

The four formulas (1), (2), (3), (4) contain all that is essential in the study of radicals. In fact, we have already seen in Chapter IX how (1), (2), and (3) are thus used. In the present chapter we shall review and extend those studies, making use now of (4) also.

127. Simplification of Radicals.

EXAMPLE 1. Simplify $\sqrt{75}$.

SOLUTION. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$. *Ans.*

(Formula (2), § 126)

EXAMPLE 2. Simplify $\sqrt[4]{\dfrac{64\,a^6b^7}{x^5}}$.

SOLUTION. $$\sqrt[4]{\frac{64\,a^6b^7}{x^5}} = \frac{\sqrt[4]{64\,a^6b^7}}{\sqrt[4]{x^5}} = \frac{\sqrt[4]{16\,a^4b^4} \times \sqrt[4]{4\,a^2b^3}}{\sqrt[4]{x^4} \times \sqrt[4]{x}}$$
$$= \frac{2\,ab\sqrt[4]{4\,a^2b^3}}{x\sqrt[4]{x}}. \quad \textit{Ans.}$$

EXERCISES

Before undertaking the following exercises, review § 44, including the note on page 72. These resemble the exercises on pages 73, 74, but in some instances are more difficult because they refer to radicals of as high orders as the fifth, sixth, and seventh.

Simplify each of the following expressions.

1. $\sqrt{52}$.
2. $\sqrt{80}$.
3. $\sqrt{72}$.
4. $\sqrt{\frac{8}{75}}$.
5. $\sqrt[3]{24}$.
6. $\sqrt[3]{72}$.
7. $\sqrt[3]{192}$.
8. $\sqrt[3]{\frac{16}{81}}$.
9. $\sqrt[4]{64}$.
10. $\sqrt[4]{243}$.
11. $\sqrt[4]{1250}$.
12. $\sqrt[4]{\frac{32}{405}}$.
13. $\sqrt[5]{64}$.
14. $\sqrt[5]{486}$.
15. $\sqrt[6]{128}$.
16. $\sqrt[7]{256}$.
17. $\sqrt{99\,a^2}$.
18. $\sqrt{60\,x^2y^4}$.
19. $\sqrt{75\,p^3q^2r}$.
20. $\sqrt[3]{128\,mn^4}$.
21. $\sqrt[3]{108\,r^4s^3t^2}$.
22. $\sqrt[4]{64\,a^5b^{10}}$.
23. $\sqrt[5]{64\,m^7n^6}$.
24. $\sqrt[6]{128\,x^6y^7}$.
25. $\sqrt[7]{128\,m^8n^7q}$.
26. $\sqrt{(a^2-b^2)(a-b)}$. $(a-b)\sqrt{a+b}$. *Ans.*
27. $\sqrt{(x^2-4\,y^2)(x+2\,y)}$.
28. $\sqrt{4\,a^5b^2+8\,a^4b^3+4\,a^3b^4}$. [HINT. See Note in § 45.]
29. $\sqrt[3]{\frac{3}{8}}$. $\frac{\sqrt[3]{3}}{2}$. *Ans.*
30. $\sqrt[3]{\frac{40}{81}}$.
31. $\sqrt[3]{\frac{2}{27\,x^6}}$.
32. $\sqrt[3]{\frac{4\,x^2}{125\,y^3}}$.
33. $\sqrt[4]{\frac{5\,a}{16\,b^4}}$.
34. $\sqrt[3]{\frac{7\,g}{81\,h^5}}$.
35. $\sqrt[5]{\frac{4\,x}{32\,y^6}}$.
36. $\sqrt[6]{\frac{3\,x^3}{64\,y^6}}$.
37. $\sqrt[7]{\frac{7\,m^{14}}{128\,x^9y^{14}}}$.

EXERCISES

Write each of the following in a form having no coefficient outside the radical sign. First review the similar exercises on page 74.

1. $3\sqrt[3]{2}$.

SOLUTION. $3\sqrt[3]{2}=\sqrt[3]{27}\times\sqrt[3]{2}=\sqrt[3]{27\times 2}=\sqrt[3]{54}$. *Ans.*

2. $2\sqrt[3]{3}$.
3. $2\sqrt[4]{2}$.
4. $3\sqrt[4]{3}$.
5. $2\sqrt[5]{7}$.
6. $3\sqrt[6]{6}$.
7. $4a\sqrt{2a}$.
8. $6x^2\sqrt{5x^3}$.
9. $2ab\sqrt[3]{5a^2}$.
10. $m\sqrt[4]{2n}$.
11. $2x\sqrt[5]{xy}$.
12. $(x+y)\sqrt{x-y}$.
13. $\dfrac{2x}{3}\sqrt{\dfrac{5y}{2x}}$.
14. $\dfrac{2x}{3}\sqrt[3]{\dfrac{5y}{2x}}$.

128. Reduction of Radicals of Different Orders to Equivalent Radicals of the Same Order.

EXAMPLE. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same order.

SOLUTION. By use of Formula (4), § 126, and the laws of exponents (§ 117) we may write

$$\sqrt{2}=2^{1/2}=2^{6/12}=\sqrt[12]{2^6}=\sqrt[12]{64},$$
$$\sqrt[3]{3}=3^{1/3}=3^{4/12}=\sqrt[12]{3^4}=\sqrt[12]{81},$$
$$\sqrt[4]{5}=5^{1/4}=5^{3/12}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

NOTE. As now expressed, the given radicals may be compared as to their magnitudes. Thus we see $\sqrt[4]{5}$ is the greatest of the three since (the orders of the radicals being now the same) it has the largest radicand, namely, 125.

An examination of the process just followed leads to the following rule.

To reduce radicals to equivalent radicals of the same order:

1. *Express the radicals with fractional exponents.*
2. *Reduce the exponents to a common denominator.*
3. *Rewrite the results thus obtained in radical form.*

EXERCISES

Reduce each of the following groups to equivalent radicals of the same order.

1. $\sqrt{3}$ and $\sqrt[3]{4}$.
2. $\sqrt{2}$ and $\sqrt[5]{3}$.
3. $\sqrt[3]{3}$ and $\sqrt[5]{5}$.
4. $\sqrt{5}$, $\sqrt[3]{6}$, and $\sqrt[4]{7}$.
5. $\sqrt{a}$ and $\sqrt[3]{b}$. $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$. *Ans.*
6. $\sqrt{2x}$ and $\sqrt[3]{3x}$.
7. $\sqrt{xy}$, $\sqrt[3]{yz}$, and $\sqrt[4]{xz}$.
8. $\sqrt{1-x}$, $\sqrt[3]{1+x}$.
9. $\sqrt[3]{(a+b)^2}$, $\sqrt[4]{(a+b)^3}$, $\sqrt[5]{(a+b)^4}$.
10. $\sqrt{\frac{a}{b}}$, $\sqrt[3]{\frac{a^2}{b^2}}$, $\sqrt[4]{\frac{a^3}{b^3}}$, $\sqrt[5]{\frac{a^4}{b^4}}$.

Arrange the following in order of magnitude. (See § 128.)

11. $\sqrt{3}$, $\sqrt[3]{4}$.
12. $\sqrt[3]{2}$, $\sqrt[4]{3}$.
13. $\sqrt[5]{10}$, $\sqrt[3]{5}$.
14. $\sqrt{4}$, $\sqrt[5]{13}$.
15. $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[4]{6}$.
16. $\sqrt[3]{13}$, $\sqrt{7}$, $\sqrt[6]{174}$.

129. Multiplication of Radicals. We have already seen in § 46 how to multiply two or more radicals of the *second* order. Radicals of higher order than the second may be multiplied by means of the general formula (2) of § 126.

EXAMPLE 1. $\sqrt[3]{4} \cdot \sqrt[3]{10} = \sqrt[3]{40}$. (Formula (2), § 126.) Simplifying (§ 127), $\sqrt[3]{40} = 2\sqrt[3]{5}$. *Ans.*

EXAMPLE 2.
$$\sqrt{4x} \cdot \sqrt[3]{2x^2} = \sqrt[6]{(4x)^3} \cdot \sqrt[6]{(2x^2)^2}$$
$$= \sqrt[6]{(4x)^3 \cdot (2x^2)^2}. \quad \text{(Formula (2), § 126)}$$

But
$$\sqrt[6]{(4x)^3(2x^2)^2} = \sqrt[6]{4^3 \cdot 2^2 \cdot x^3 \cdot x^4} = \sqrt[6]{2^6 \cdot 2^2 \cdot x^7}$$
$$= \sqrt[6]{(2x)^6 \cdot (4x)} = 2x\sqrt[6]{4x}. \quad \textit{Ans.}$$

Note that in Ex. 2 the given radicals are of *different* orders, in which case the first step is to reduce them to equivalent radicals of the same order (§ 128). Then apply (2) of § 126.

In general, we have the following rule.

To multiply radicals of any orders:

1. *Reduce the radicals, if necessary, to equivalent radicals of the same order* (§ 128).

2. *Multiply the resulting radicals by use of Formula* (2), § 126.

3. *Simplify the result as in* §§ 44 *and* 127.

EXERCISES

[Compare with the exercises on page 76.]

Find each of the following indicated products.

1. $\sqrt[3]{9} \cdot \sqrt[3]{3}$.
2. $\sqrt[3]{6} \cdot \sqrt[3]{4}$.
3. $2\sqrt[3]{18} \cdot \sqrt[3]{3}$.
4. $2\sqrt[4]{4} \cdot \sqrt[4]{12}$.
5. $3\sqrt[5]{24} \cdot \sqrt[5]{4}$.
6. $\sqrt[3]{9x^2} \cdot \sqrt[3]{3x^2}$.
7. $x\sqrt[4]{6x^2} \cdot \sqrt[4]{8x^3}$.
8. $5\sqrt[5]{x^6y^4} \cdot \sqrt[5]{xy^5}$.
9. $\sqrt{2} \cdot \sqrt[3]{4}$.
10. $\sqrt[3]{2} \cdot \sqrt[6]{8}$.
11. $\sqrt{10} \cdot \sqrt[4]{4}$.
12. $\sqrt{x} \cdot \sqrt[3]{x}$.
13. $\sqrt{y} \cdot \sqrt[4]{y^3}$.
14. $\sqrt[3]{x^2} \cdot \sqrt[6]{y}$.
15. $\sqrt{2} \cdot \sqrt[3]{3} \cdot \sqrt[4]{4}$.
16. $\sqrt[3]{x^2y^2z} \cdot \sqrt[9]{x^3yz^2}$.

130. Division of Radicals. We have already seen in § 47 how one radical of the *second* order may be divided by another of that order. If the radicals are of higher order than the second, we may divide them by means of the general formula (3) of § 126.

EXAMPLE 1. Divide $\sqrt[3]{16}$ by $\sqrt[3]{12}$.

SOLUTION. $\dfrac{\sqrt[3]{16}}{\sqrt[3]{12}} = \sqrt[3]{\dfrac{16}{12}} = \sqrt[3]{\dfrac{4}{3}}$. (Formula (3), § 126)

The simplest and most desirable form in which to leave this result is that in which no fraction appears except *outside* the sign. Thus

$$\sqrt[3]{\frac{4}{3}} = \sqrt[3]{\frac{4 \times 3^2}{3^3}} = \frac{\sqrt[3]{36}}{\sqrt[3]{3^3}} = \frac{1}{3}\sqrt[3]{36}. \quad \textit{Ans.}$$

EXAMPLE 2. Divide $\sqrt{3}$ by $\sqrt[3]{9}$.

SOLUTION. $\dfrac{\sqrt{3}}{\sqrt[3]{9}}=\dfrac{\sqrt[6]{3^3}}{\sqrt[6]{9^2}}=\sqrt[6]{\dfrac{27}{81}}=\sqrt[6]{\dfrac{1}{3}}.$ (Formula (3), § 126)

But $\sqrt[6]{\dfrac{1}{3}}=\sqrt[6]{\dfrac{3^5}{3^6}}=\dfrac{\sqrt[6]{243}}{\sqrt[6]{3^6}}=\dfrac{1}{3}\sqrt[6]{243}.$ *Ans.*

EXAMPLE 3. Divide $\sqrt[3]{3\,xy}$ by $\sqrt[6]{3\,x^2y^3}$.

SOLUTION. $\dfrac{\sqrt[3]{3\,xy}}{\sqrt[6]{3\,x^2y^3}}=\dfrac{\sqrt[6]{(3\,xy)^2}}{\sqrt[6]{3\,x^2y^3}}=\sqrt[6]{\dfrac{9\,x^2y^2}{3\,x^2y^3}}=\sqrt[6]{\dfrac{3}{y}}.$

But $\sqrt[6]{\dfrac{3}{y}}=\sqrt[6]{\dfrac{3\,y^5}{y^6}}=\dfrac{\sqrt[6]{3\,y^5}}{\sqrt[6]{y^6}}=\dfrac{1}{y}\sqrt[6]{3\,y^5}.$ *Ans.*

In general, we have the following rule.

To divide one radical by another:

1. *Reduce the radicals, if necessary, to equivalent radicals of the same order* (§ 128).
2. *Divide the resulting radicals by use of Formula* (3) (§ 126).
3. *Simplify the result in such a way that no fraction appears except outside the radical sign.*

EXERCISES

[Compare with the exercises on page 77.]

Find each of the following indicated quotients.

1. $\sqrt{7}\div\sqrt{2}$.
2. $\sqrt[3]{7}\div\sqrt[3]{2}$.
3. $\sqrt{7}\div\sqrt[3]{2}$.
4. $2\div\sqrt[3]{2}$.
5. $3\div\sqrt[4]{3}$.
6. $\sqrt{\frac{2}{3}}\div\sqrt[3]{\frac{2}{3}}$.
7. $\sqrt[3]{12\,x^2}\div\sqrt{2\,x}$.
8. $\sqrt[3]{81\,x^2y}\div\sqrt[6]{9\,xy}$.
9. $\sqrt{8\,m}\div\sqrt[4]{32\,m}$.
10. $\sqrt[3]{ax}\div\sqrt{xy}$.
11. $\sqrt{2\,mn^3}\div\sqrt[4]{m^4n^4}$.
12. $\sqrt[3]{a^2x^2}\div\sqrt{2\,ax}$.
13. $3\sqrt{75}\div5\sqrt{28}$.
14. $\sqrt[3]{16}\div\sqrt[6]{32}$.
15. $\sqrt{a-b}\div\sqrt{a+b}$.

131. Involution and Evolution of Radicals. By use of Formula (4) of § 126 together with the general laws of exponents (§ 117), we may raise a radical to a power or extract a root of it.

EXAMPLE 1. Find the square of $\sqrt[4]{8}$.

SOLUTION. $(\sqrt[4]{8})^2 = (8^{1/4})^2 = 8^{2/4} = 8^{1/2} = \sqrt{8} = 2\sqrt{2}$. *Ans.*

EXAMPLE 2. Find the fourth root of $\sqrt{2x}$.

SOLUTION. $\sqrt[4]{\sqrt{2x}} = [(2x)^{1/2}]^{1/4} = (2x)^{1/8} = \sqrt[8]{2x}$. *Ans.*

EXERCISES

Perform each of the following indicated involutions.

1. $(\sqrt[4]{3})^2$.	6. $(2\sqrt{3})^3$.	11. $(\sqrt{2})^4$.
2. $(\sqrt[6]{8})^2$.	7. $(3\sqrt{2})^3$.	12. $(3\sqrt{2})^3$.
3. $(3\sqrt{x})^2$.	8. $(2\sqrt[3]{a^2})^3$.	13. $(2\sqrt{2xy})^4$.
4. $(2\sqrt[3]{3x})^2$.	9. $(\sqrt[4]{4m^2n^3})^3$.	14. $(-\sqrt{2}\sqrt[3]{ay^2})^4$.
5. $(x^2\sqrt{4y})^2$.	10. $(\sqrt[6]{9x^3})^3$.	15. $(\sqrt{x}\sqrt[3]{y})^6$.

Perform each of the following indicated evolutions.

16. $\sqrt{\sqrt{2}}$.	20. $\sqrt{\sqrt[5]{x^{12}}}$.	24. $\sqrt[3]{\sqrt{24}}$.
17. $\sqrt{\sqrt[3]{5}}$.	21. $\sqrt{\sqrt{81m^5n^6}}$.	25. $\sqrt[3]{\sqrt{7a^3}}$.
18. $\sqrt{\sqrt[6]{x^2}}$.	22. $\sqrt[3]{\sqrt{27}}$.	26. $\sqrt[5]{\sqrt{32}}$.
19. $\sqrt{\sqrt[5]{49}}$.	23. $\sqrt{\sqrt[3]{25}}$.	27. $\sqrt[5]{\sqrt[4]{2xy^3}}$.

132. Rationalizing the Denominator of a Fraction. If the denominator of a fraction consists of a single quadratic radical, or is a binomial containing quadratic radicals, the fraction may be changed into one which has radicals only in its numerator. The process of doing this is called ***rationalizing the denominator***.

EXAMPLE 1. Rationalize the denominator in the fraction

$$\frac{\sqrt{3}}{\sqrt{5}}.$$

SOLUTION. Here the denominator contains the single surd $\sqrt{5}$. To rationalize this denominator it is merely necessary to multiply both numerator and denominator by $\sqrt{5}$, giving $\sqrt{15}/5$. *Ans.*

EXAMPLE 2. Rationalize the denominator in the fraction

$$\frac{1}{\sqrt{3}+\sqrt{2}}.$$

SOLUTION. Multiply both numerator and denominator by $\sqrt{3}-\sqrt{2}$, giving

$$\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}=\frac{\sqrt{3}-\sqrt{2}}{3-2}=\frac{\sqrt{3}-\sqrt{2}}{1}$$
$$=\sqrt{3}-\sqrt{2}. \quad \textit{Ans.}$$

EXAMPLE 3. Rationalize the denominator in the fraction

$$\frac{3\sqrt{5}+2\sqrt{2}}{\sqrt{5}-\sqrt{2}}.$$

SOLUTION. Multiplying both numerator and denominator by $\sqrt{5}+\sqrt{2}$, we have

$$\frac{(3\sqrt{5}+2\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}=\frac{3\cdot 5+3\sqrt{10}+2\sqrt{10}+2\cdot 2}{5-2}$$
$$=\frac{19+5\sqrt{10}}{3}. \quad \textit{Ans.}$$

We may then state the following rule.

To rationalize the denominator of a fraction:

If the denominator contains a single radical, multiply both numerator and denominator by that radical.

If the denominator has either of the binomial forms $\sqrt{a}+\sqrt{b}$ *or* $\sqrt{a}-\sqrt{b}$, *multiply both numerator and denominator by* $\sqrt{a}-\sqrt{b}$, *or* $\sqrt{a}+\sqrt{b}$ *according as we have the first or second of these two cases.*

EXERCISES

Rationalize the denominators in each of the following fractions.

1. $\dfrac{\sqrt{2}}{\sqrt{5}}$.

2. $\dfrac{\sqrt{8}}{\sqrt{10}}$.

3. $\dfrac{\sqrt{a}}{\sqrt{b}}$.

4. $\dfrac{\sqrt{2\,ax}}{\sqrt{3\,bx}}$.

5. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

6. $\dfrac{1}{\sqrt{3}-1}$.

7. $\dfrac{2}{\sqrt{7}-\sqrt{2}}$.

8. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}$.

9. $\dfrac{3\sqrt{3}-2\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$.

10. $\dfrac{3+\sqrt{6}}{2\sqrt{5}}$.

11. $\dfrac{2\sqrt{a}-3\sqrt{b}}{3\sqrt{a}-2\sqrt{b}}$.

12. $\dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+1}$.

133. Finding the Value of Fractions Containing Radicals. Suppose we wish to find the value of

$$\frac{1}{\sqrt{3}+\sqrt{2}}$$

correct to five places of decimals. It is well to begin by rationalizing the denominator, thus making the fraction take the form (see Ex. 2 worked in § 132) $\sqrt{3}-\sqrt{2}$. All we now need to do is to look up in the table the values of $\sqrt{3}$ and $\sqrt{2}$ so as to work out the value of $\sqrt{3}-\sqrt{2}$. That is, we have

$$\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{3}-\sqrt{2}=1.73205^{+}-1.41421^{+}=0.31784^{+}. \quad \textit{Ans.}$$

If, in this example, we had not first rationalized the denominator, we should have had to find the value of

$$\frac{1}{1.73205^{+}+1.41421^{+}}, \quad \text{or} \quad \frac{1}{3.14626^{+}},$$

which would compel us to *divide* 1 by 3.14626. Note how much more difficult this is than the above, where virtually all we need to do is to *subtract* 1.41421 from 1.73205.

This illustrates the general fact that *to find the value of a fraction, its denominator (if it contains radicals) should first be rationalized whenever possible.*

EXERCISES

Find (by first rationalizing the denominator and then using the table) the approximate values of the following fractions.

1. $\dfrac{2}{\sqrt{3}}$.
2. $\dfrac{2\sqrt{7}}{3\sqrt{5}}$.
3. $\dfrac{1}{\sqrt{3}-\sqrt{2}}$.
4. $\dfrac{1+\sqrt{3}}{2-\sqrt{3}}$.
5. $\dfrac{1}{\sqrt{250}}$.
6. $\dfrac{2\sqrt{5}-4}{3\sqrt{3}-2}$.

134. Binomial Surd.** A binomial, one or both of whose terms are surds (§ 42), is called a ***binomial surd.

Thus $2+\sqrt{5}$, $\sqrt{2}+\sqrt{5}$, $\sqrt{3}-1$ are binomial surds.

***135. To Find the Square Root of a Binomial Surd.** The familiar formula for $(a+b)^2$ (§ 10, Formula VI) may be put into the form

(1) $$a^2+b^2+2\,ab=(a+b)^2.$$

Since this relation holds true for any values of a and b, let us suppose in particular that both are positive, in which case we may write $a=\sqrt{x}$ and $b=\sqrt{y}$, where x and y are properly chosen positive values. The equation just written then takes the form

$$x+y+2\sqrt{xy}=(\sqrt{x}+\sqrt{y})^2.$$

Extracting the square root of both members now gives

(2) $$\sqrt{(x+y)+2\sqrt{xy}}=\sqrt{x}+\sqrt{y}.$$

This formula, having been thus derived from (1), must therefore hold true for any positive values of x and y.

Similarly, by starting with the familiar formula for $(a-b)^2$, we arrive at the formula

(3) $$\sqrt{(x+y)-2\sqrt{xy}}=\sqrt{x}-\sqrt{y}.$$

Formulas (2) and (3) are frequently used to obtain the square root of a binomial surd (§ 134) as illustrated in the following example.

EXAMPLE. Find the square root of the binomial surd $11+4\sqrt{7}$.

SOLUTION. We are to find $\sqrt{11+4\sqrt{7}}$.

This may be written $\sqrt{11+2\sqrt{28}}$, and it is now in the form of the first member of Formula (2), provided we choose x and y so that $x+y=11$ while $xy=28$.

The values of x and y which satisfy these last two equations are seen (by inspection) to be $x=4$, $y=7$.

Substituting these values of x and y in the second member of (2) gives $\sqrt{4}+\sqrt{7}=2+\sqrt{7}$.

Therefore $$\sqrt{11+4\sqrt{7}}=2+\sqrt{7}. \quad \textit{Ans.}$$

CHECK.

$$(2+\sqrt{7})^2=2^2+2\cdot 2\cdot \sqrt{7}+(\sqrt{7})^2=4+4\sqrt{7}+7=11+4\sqrt{7}.$$

The pupil will observe that all that is essential in working the above example is to write the given surd term ($4\sqrt{7}$) so that it has the coefficient 2 instead of 4. Similarly, all such problems may be brought under Formulas (2) or (3) as soon as the coefficient of the surd term has been reduced to 2.

* EXERCISES

Find the square root of each of the following expressions, and check your answer.

1. $6+2\sqrt{8}$.

[HINT. Here $x+y=6$, $xy=8$.]

2. $6-2\sqrt{8}$. **4.** $11-2\sqrt{30}$. **6.** $6+\sqrt{32}$. **8.** $8+4\sqrt{3}$.

3. $7+4\sqrt{3}$. **5.** $6-\sqrt{20}$. **7.** $7-\sqrt{40}$. **9.** $20-6\sqrt{11}$.

10. Establish Formula (3) of § 135 by a process similar to that used in establishing Formula (2).

CHAPTER XX

LOGARITHMS

I. General Considerations †

136. Definition of Logarithms. If we ask what power of 10 must be used to give a result of 100, the answer is 2 because $10^2 = 100$. Another common way of stating this is to say that "the *logarithm* of 100 is 2." In the same way, the power of 10 needed to give 1000 is 3 because $10^3 = 1000$, and this is briefly stated by saying that "the *logarithm* of 1000 is 3." Similarly, the power of 10 that gives .1 is -1 because $10^{-1} = \frac{1}{10}$, or .1 (§ 123), and this is equivalent to saying that "the *logarithm* of .1 is -1." Likewise, the logarithm of .01 is -2. Why?

From these illustrations we readily see what is meant by the logarithm of a number. It may be defined as follows:

The ***logarithm*** *of a number is the power of* 10 *required to give that number.*

Note. A more general definition will be given in § 151, but this is the one commonly used in practice.

The fact that the logarithm of 100 is 2 is written $\log 100 = 2$. Similarly, we have $\log 1000 = 3$, $\log .1 = -1$, $\log .01 = -2$, etc.

† Parts I and II give definitions and essential theorems which should be well understood before Part III, which describes the important applications, is taken up.

EXERCISES

1. What is the *meaning* of log 10000? What is its *value?*

2. What is the value of log .001? Why?

3. What is the value of log .00001? Why?

4. What is the value of log 10?

5. What is the value of log 1? (See § 122.)

6. As a number increases from 100 to 1000 how does its logarithm change?

7. As a number decreases from .1 to .01 how does its logarithm change? Answer the same as the number goes from .01 to .001; from 1 to 10; from 1 to 1000.

8. Explain why the following are true statements:

(*a*) $\log 100000 = 5$. (*b*) $\log .0001 = -4$.

(*c*) $\log \sqrt{10} = \frac{1}{2}$.

[Hint. Remember $\sqrt{10} = 10^{1/2}$.

(*d*) $\log \sqrt[3]{10} = \frac{1}{3}$.

(*e*) $\log \sqrt[3]{100} = \frac{2}{3}$.

[Hint. Remember $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{2/3}$. (§ 121.)]

(*f*) $\log \sqrt{.1} = -\frac{1}{2}$.

137. Logarithm of Any Number. Suppose we ask what the value is of log 236. What we are asking for (see definition in § 136) is that value which, when used as an exponent to 10, will give 236; that is we wish the value of x which will satisfy the equation $10^x = 236$. This question resembles those in § 136, but is different because we cannot immediately arrive at the desired value of x *by mere inspection.* All we can say here at the beginning is that x must lie *somewhere* between 2 and 3, because $10^2 = 100$ and $10^3 = 1000$, and 236 lies between these two numbers. In order to find x to a finer degree of accuracy, it is now natural to try for it such values

as 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, and 2.9, all of which lie between 2 and 3. The result (which for brevity we shall here state without proof) is that when $x=2.3$ the value of 10^x is slightly *less* than our given number 236, while if we take $x=2.4$ the value of 10^x is slightly *greater* than 236. Thus x lies somewhere between 2.3 and 2.4. In other words, the value of log 236 *correct to the first decimal place* (compare § 37) is 2.3.

It is now natural, if we wish to obtain x to still greater accuracy, to try for it such values as 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, and 2.39, all of which lie between 2.3 and 2.4. The result (which again is here stated without proof) is that when $x=2.37$ the value of 10^x is slightly *less* than our number 236, while if we take $x=2.38$ the value of 10^x is slightly *greater* than 236. This means that the second figure of the decimal is 7, after which we may say that the value of log 236 *correct to two places of decimals* is 2.37.

Proceeding farther in the same manner, it can be shown that when $x=2.372$ the value of 10^x is slightly less than 236, while for $x=2.373$ the value of 10^x is slightly greater than 236. Thus the value of log 236 *correct to three places of decimals* is 2.372. Similarly, it can be shown that the number in the fourth decimal place is 9, and this is as far as it is necessary to carry out the process, since the result is then sufficiently accurate for all ordinary purposes.

In summary, then, we have $\log 236=2.3729$, this value being correct to four places of decimals.

NOTE. It thus appears that logarithms do not in general come out *exact*, though they do so for such exceptional numbers as 100, 1000, 10,000, .1, .01, etc. (Compare § 37.) They can be expressed only approximately, yet as accurately as one pleases by carrying out the decimal far enough. In this respect they resemble such numbers as $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[4]{3}$, etc.

Other examples of logarithms are given below. Note especially the decimal part of each, which is correct to four places.

$\log 283 = 2.4518$	$\log 196 = 2.2923$	$\log 17 = 1.2304$
$\log 6 = 0.7782$	$\log 3.410 = 0.5328$	$\log 5.75 = 0.7597$

138. Characteristic. Mantissa. We have seen that the logarithm of a number consists (in general) of an integral part and a decimal part.

Thus $\log 236 = 2.3729$. Here the integral part is 2 and the decimal part is .3729. Similarly, in $\log 6 = 0.7782$ the integral part is 0, while the decimal part is .7782.

These two parts of every logarithm are given special names as follows:

The integral part of a logarithm is called the ***characteristic*** *of the logarithm.*

The decimal part of a logarithm is called the ***mantissa*** *of the logarithm.*

Thus the characteristic of log 236 is 2, while its mantissa is .3729. (See above.) Similarly, the characteristic of log 6 is 0, while its mantissa is .7782.

EXERCISES

1. What is the characteristic of log 100? What the mantissa? Answer the same questions for log 1000, log 10, and log 1.

2. What is the characteristic of log 156?

[Hint. Note that 156 lies between 10^2 and 10^3.]

3. What is the characteristic of log 276? of log 1376? of log 97? of log 18? of log 5? of log 11? of log 14798?

4. For what kind of number can one tell *by inspection* both the characteristic and the mantissa of its logarithm? (See § 136.)

139. Further Study of Characteristic and Mantissa. We have seen (§ 138) that log 236 = 2.3729, which is the same as saying that

(1) $$10^{2.3729} = 236.$$

Let us now multiply both members of (1) by 10. The left side becomes $10^{2.3729+1}$ or $10^{3.3729}$ (§ 117, Law I) while the right side becomes 2360. That is, we have $10^{3.3729} = 2360$, which is the same as saying that

$$\log 2360 = 3.3729$$

If, instead of multiplying both sides of (1) by 10, we *divide* both by 10, we obtain in like manner $10^{2.3729-1} = 23.6$ (§ 117, Law II). That is, we have $10^{1.3729} = 23.6$, which is the same as saying that

$$\log 23.6 = 1.3729$$

Finally, if we divide both sides of (1) by 10^2, or 100, we obtain $10^{2.3729-2} = 2.36$. That is, we have $10^{0.3729} = 2.36$ which is the same as saying that

$$\log 2.36 = 0.3729$$

What we now wish to do is to *compare* the results which we have just been obtaining, and for this purpose they are arranged side by side in a column below.

(2) $$\begin{cases} \log\ 2360 = 3.3729 \\ \log\ 236 = 2.3729 \\ \log\ 23.6 = 1.3729 \\ \log\ 2.36 = 0.3729 \end{cases}$$

Note that the mantissas here appearing on the right are all the same, namely .3729, while the numbers appearing on the left (that is, 2360, 236, 23.6, and 2.36) are alike *except* for the position of the decimal point, that is they contain the same significant figures. This illustrates the following important rule.

RULE I. *If two or more numbers have the same significant figures (that is, differ only in the location of the decimal point), their logarithms will have the* **same** *mantissas, that is their logarithms can differ only in their characteristics.*

Thus log 243, log 2430, log 24.3, log 2.43, log .243, and log .0243 all have the *same* mantissas. It is only their characteristics that can be different.

EXERCISES

Apply Rule I, § 139, to tell which of the following logarithms have the same mantissas.

log .167	log 8100	log 16.7	log 81	log .0072
log .081	log 7.2	log 720	log 1670	log 16700

II. To Determine the Logarithm of Any Number

140. Purpose of This Part. When we wish to determine the value of a logarithm, as, for example, to find log 236, we can work out the characteristic and mantissa as explained in § 137, but this requires considerable time. What we do *in practice* is to use certain simple rules for determining the characteristic, and we determine the mantissa directly from certain tables which have been carefully prepared for the purpose. We shall now state these rules (§§ 141–143) and explain the tables and how to use them (§§ 144–146).

141. Characteristics for Numbers Greater than 1. If we look again at the results in (2) of § 139, we see that the characteristic of log 2360 is 3. Thus the characteristic is 1 less than the number of figures to the *left* of the decimal point.

NOTE. 2360 is the same as 2360., so that there are *four* figures here to the left of the decimal point.

Again, we see from (2) of § 139 that the characteristic of log 236 is 2 and this, as in the case already examined, is 1 less than the number of figures to the left of the decimal point.

NOTE. 236 is the same as 236., so there are *three* figures here to the left of the decimal point.

Similarly, since the characteristic of log 23.6 is 1 (see (2) of § 139) this again obeys the same law as just observed in the other two cases, that is, the characteristic is 1 less than the number of figures to the left of the decimal point.

Finally, since the characteristic of log 2.36 is 0, the same law is again present here. Explain.

The law which we have just observed can be shown in like manner to hold good for the characteristic of the logarithm of any number greater than 1; hence we may state the following general rule.

RULE II. *The characteristic of the logarithm of a number greater than* 1 *is one less than the number of figures to the left of the decimal point.*

Thus the characteristic of log 385.9 is 2; that of log 8.679 is 0.

EXERCISES

State, by Rule II, § 141, the characteristic of the logarithm of each of the following numbers.

1. 476.5	**5.** 89.65	**9.** 500.005
2. 325.	**6.** 105,000.	**10.** 3076.8
3. 8976.	**7.** 17.694	**11.** 41.
4. 1.6	**8.** 2.0815	**12.** 3.25679

State how many figures precede the decimal point of a number if the characteristic of its logarithm is

13. 3. **14.** 2. **15.** 0. **16.** 1. **17.** 4. **18.** 5.

142. Characteristics for Positive Numbers Less Than 1. We have seen (see (2) in § 139) that log 2.36 = 0.3729, which is the same as saying that

(1) $$10^{0.3729} = 2.36$$

Let us now divide both members of this relation by 10. We thus obtain (§ 117, Law II)

$$10^{0.3729-1} = .236 \quad (\text{or } 10^{-1+0.3729} = .236),$$

which gives us (by § 136)

$$\log .236 = -1 + 0.3729$$

Observe that $-1+0.3729$ is really a negative quantity, being equal to $-(1-0.3729)$ which reduces to -0.6271. However, it is more convenient for our present purposes to keep the longer form $-1+0.3729$. Note that this *cannot* be written as -1.3729 because this last is equal to $-1-0.3729$ instead of $-1+0.3729$.

If, instead of dividing both members of (1) by 10, we divide both by 10^2, or 100, we obtain

$$10^{0.3729-2} = .0236 \quad (\text{or } 10^{-2+0.3729} = .0236),$$

which means that

$$\log .0236 = -2 + 0.3729$$

Similarly, by dividing (1) by 10^3, or 1000, we find that

$$\log .00236 = -3 + 0.3729$$

Finally, if we divide (1) by 10^4, or 10000, we find that

$$\log .000236 = -4 + 0.3729$$

Let us now compare the four results just obtained. Beginning with the last result, we see that in the number .000236 there are *three* zeros immediately to the *right* of the decimal point, that is, between the decimal point and the first significant figure. Corresponding to this, the characteristic on the right is *minus four*. Hence the characteristic is negative and 1 more numerically than the number of zeros between the decimal point and the first significant figure.

Similarly, in the number .00236 there are *two* zeros between the decimal point and the first significant figure, and corresponding to this there is a characteristic on the right of *minus three.* Hence, as before, the characteristic here is negative and numerically 1 more than the number of zeros between the decimal point and the first significant figure. This statement, which is true in all cases mentioned above, can be proved for the characteristic of the logarithm of any positive number less than 1. Hence we have the following rule.

RULE III. *The characteristic of the logarithm of a (positive) number less than* 1, *is negative, and is numerically* 1 *greater than the number of zeros between the decimal point and the first significant figure.*

Thus the characteristic of log .0076 is -3; that of log .28 is -1.

NOTE. The logarithm of a negative number is an imaginary quantity (as shown in higher mathematics), and hence we shall consider here the logarithms of positive numbers only.

143. Usual Method of Writing a Negative Characteristic. In § 142 we saw that $\log .236 = -1 + 0.3729$. If we add 10 to this quantity and at the same time subtract 10 from it, we do not change its value, but we give it the new form $9 + 0.3729 - 10$, which is the same as $9.3729 - 10$. That is, we may write

$$\log .236 = 9.3729 - 10.$$

This is the form used in practice.

Likewise, instead of writing $\log .0236 = -2 + 0.3729$ (see § 142) we write in practice

$$\log .0236 = 8.3729 - 10,$$

and similarly we write

$$\log .00236 = 7.3729 - 10.$$

Thus the usual method of expressing the characteristic of -1 is to write $9-10$ for it; if it is -2, we write $8-10$ for it; if it is -3, we write $7-10$ for it, etc.

For example, log .0076 has the characteristic $7-10$.

EXERCISES

State, by Rule III, § 142, the value of the characteristic of the logarithm of each of the following; state how it would be written if expressed in the usual form described in § 143.

1. .06 -2, or $8-10$. *Ans.*

2. .0087	**5.** .0835	**8.** .00978
3. .75	**6.** .835	**9.** .12345
4. .00067	**7.** .33764	

How many zeros lie between the decimal point and the first significant figure of a number when the characteristic of its logarithm is

10. -3. **11.** $9-10$. **12.** -5. **13.** $8-10$. **14.** $7-10$.

144. Determination of Mantissas. Use of Tables. Suppose we wish to determine completely the value of log 187. By Rule II, § 141, we know that the characteristic is 2. To find the mantissa, we turn to the tables (p. 226) and look in the column headed **N** for the first two figures of the given number, that is, for 18. The desired mantissa is then to be found on the horizontal line with these two figures and in the column headed by the third figure of the given number, that is, in the column headed by 7. Thus in the present case the mantissa is found to be .2718.

NOTE. For brevity, the decimal point preceding each mantissa is omitted from the tables. It must be supplied as soon as t e mantissa is used.

The complete value (correct to four decimal places) of log 187 is therefore 2.2718.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Again, suppose we wish to determine log 27.6. The characteristic (by § 141) is 1. The mantissa, by Rule I, § 139, is the same as that of log 276; and the latter, as given in the tables, is .4409. Therefore, log $27.6 = 1.4409$. *Ans.*

As a last example, suppose we wish to determine log .0173. The characteristic (by § 142) is -2, or 8–10. The mantissa, by the rule in § 139, is the same as that of log 173 and the latter, as obtained from the tables, is .2380. Therefore, log $.0173 = 8.2380 - 10$. *Ans.*

These examples illustrate how the tables together with Rules II and III, §§ 141, 142, enable us to determine completely the logarithm of any number provided it contains no more than three significant figures. We may now summarize our results in the following rule.

RULE IV. *To find the logarithm of a number of three significant figures:*

1. *Look in the column headed* **N** *for the first two figures of the given number. The mantissa will then be found on the horizontal line opposite these two figures and in the column headed by the third figure of the given number.*

2. *Prefix the characteristic according to Rules II and III,* §§ 141, 142.

EXERCISES

Determine the logarithm of each of the following numbers, expressing all negative characteristics as explained in § 143.

1. 451. **2.** 318. **3.** 861. **4.** 900.

5. 72.5 [HINT. Note how log 27.6 was obtained in § 144.]

6. 7.25 **7.** 93. [HINT. Write as 93.0]

8. 9. [HINT. Write as 9.00] **9.** .0136

10. .936 **11.** .0036 [HINT. Write as .00360]

12. 8560.	**15.** .45	**18.** .000235
13. .081	**16.** 61.7	**19.** $\frac{1}{2}$.
14. .8	**17.** 23,500.	**20.** $\frac{2}{3}$.

145. To Find the Logarithm of a Number of More Than Three Significant Figures. Suppose we wish to determine log 286.7. Here we have *four* significant figures while our tables only tell us the mantissas of numbers having three (or less) significant figures (as in § 144 and in the preceding exercises). In such cases we proceed as follows:

From the tables

$$\left.\begin{aligned}\log\ \ 286 &= 2.4564\\ \log 286.7 &= \ ?\\ \log\ \ 287 &= 2.4597\end{aligned}\right\}\ \textit{Difference} = 2.4597 - 2.4564 = .0033$$

Since 286.7 lies between 286 and 287, its logarithm must lie between their logarithms. Now, an increase of one unit in the number (in going from 286 to 287) produces an increase of .0033 in the mantissa. It is therefore assumed that an increase of .7 in the number (in going from 286 to 286.7) produces an increase of .7 of .0033, or .00231, in the mantissa.

Therefore log 286.7 = 2.4564 + .7 of .0033 = 2.4564 + .00231 = 2.45871, so that

log 286.7 = 2.4587 (*approximately*). *Ans.*

In practice the answer is quickly obtained as follows:

The difference between any mantissa and the next higher one in the table (neglecting the decimal point) is called a *tabular difference.* The tabular difference in this example is 4597–4564, or 33. Taking .7 of this gives 23.1, which (keeping only the first two figures) we call 23, and adding this to 4564 gives 4587. This, therefore, is the required mantissa of log 286.7, so that log 286.7 = 2.4587 (*approximately*). *Ans.*

Similarly, in finding log 286.75 the tabular difference (as before) is 33. Taking .75 of 33 gives 24.75, which (keeping only two figures) has the approximate value 25.

Hence the mantissa of log 286.75 is $4564+25=4589$. Therefore log $286.75=2.4589$. *Ans.*

Below are two examples further illustrating how the above processes are quickly carried out in practice. The student should form the habit of writing the work in this form.

EXAMPLE 1. Determine the value of log 48.731

SOLUTION. $\left.\begin{array}{l}\text{Mantissa of log } 487=6875\\ \text{Mantissa of log } 488=6884\end{array}\right\}$ Tabular difference $=9$

$$.31\times 9=2.79=3 \text{ (approximately).}$$

Hence

$$\text{mantissa of log } 48.731=6875+3=6878.$$

Therefore

$$\log 48.731=1.6878 \quad \textit{Ans.}$$

EXAMPLE 2. Determine the value of log .013403

SOLUTION. Mantissa of $134=1271$
Mantissa of $135=1303$ Tabular difference $=32$.

$$.03\times 32=.096=1 \text{ (approximately).}$$

Hence

$$\text{mantissa of log } .013403=1271+1=1272.$$

Therefore

$$\log .013403=-2+.1272=8.1272-10.$$

Ans.

NOTE. The process which we have employed for determining a mantissa when it does not actually occur in the tables is called ***interpolation.*** When examined carefully, it will be seen that the process is based upon the assumption that if a number is increased by any fractional amount of itself, the logarithm of the number will likewise be increased by the *same* fractional amount of itself. Thus, in finding the mantissa of log 286.7 at the middle of p. 229, we assumed that the increase of .7 in going from 286 to 286.7 would be accompanied by like increase of .7 in the logarithm. Such an assumption, though not *exactly* correct, is very nearly so in most cases and is therefore sufficiently accurate for all ordinary purposes.

Tables of logarithms much more extensive than those on pages 226, 227 have been prepared and are commonly used. See, for example, *The Macmillan Tables.* By means of these, any desired mantissa may usually be obtained as accurately as is necessary, directly, that is without interpolation.

EXERCISES

Obtain the logarithm of each of the following numbers.

1. 678.4	**8.** 4.806	**15.** 62.856
2. 231.3	**9.** 1.508	**16.** 541.07
3. 785.4	**10.** 3.276	**17.** 6.3478
4. 492.6	**11.** .4567	**18.** 3.1416
5. 856.8	**12.** .08346	**19.** 1.7096
6. 42.17	**13.** 856.34	**20.** .15786
7. 9.567	**14.** 243.47	**21.** .085679

146. To Find the Number Corresponding to a Given Logarithm. Thus far we have considered how to determine the logarithm of a given number, but frequently the problem is reversed, that is, it is the logarithm that is given and we wish to find the number having that logarithm. The method of doing this is the reverse of the method of §§ 144–145, and is illustrated in the following examples.

EXAMPLE 1. Find the number whose logarithm is 1.9547

SOLUTION. Locate 9547 among the mantissas in the table. Having done so, we find in the column N on the line with 9547 the figures 90. These form the first two figures of the desired number.

At the head of the column containing 9547 is 1, which is therefore the third figure of the desired number.

Hence the number sought is made up of the figures 901.

The given characteristic being 1, the number just found must be pointed off so as to have *two* figures to the left of its decimal point (Rule II, § 141).

Therefore the number is 90.1. *Ans.*

EXAMPLE 2. Find the number whose logarithm is 0.6341

SOLUTION. As in Example 1, we look among the mantissas of the table to find 6341. In this case we do not find *exactly* this mantissa, but we see that the next less mantissa appearing is 6335, while the one next greater is 6345.

The numbers corresponding to these last two mantissas are seen to be 430 and 431 respectively. Whence, if x represents the number sought, we have

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{Mantissa of } \log 430 = 6335 \\ \text{Mantissa of } \log x \quad = 6341\end{array}\right\} \text{Diff.} = 6 \\ \text{Mantissa of } \log 431 = 6345\end{array}\right\} \text{Tabular difference} = 10.$$

Since an increase of 10 in the mantissa produces an increase of 1 in the number, we assume that an increase of 6 in the mantissa will produce an increase of $\frac{6}{10}$, or .6, in the number.

Hence the number sought has the figures 4306.

Since the given characteristic is 0, the number must be 4.306 (§ 141). *Ans.*

NOTE 1. The pupil will observe that in Example 1 the given mantissa actually occurs in the tables, while in Example 2 it does not, thus making it necessary in this last case to interpolate. (See the Note in § 145.)

NOTE 2. The number whose logarithm is a given quantity is called the ***antilogarithm*** of that quantity. Thus 100 is the antilogarithm of 2, 1000 is the antilogarithm of 3, etc.

EXERCISES

Find the numbers whose logarithms are given below.

1. 1.8751	**9.** 1.4893
2. 2.9405	**10.** 2.8588
3. 0.3856	**11.** 3.7430
4. 3.5866	**12.** 0.5240
5. $9.6955-10$	**13.** 0.6970
6. $8.7152-10$	**14.** $9.7400-10$
7. $7.4900-10$	**15.** $8.3090-10$
8. $6.8519-10$	**16.** $7.5308-10$

NAPIER

(*John Napier*, 1550–1617)

Famous as the inventor of logarithms and first to show the advantage of using them in reducing the labor of ordinary computations. Interested and active also in the political and religious controversies of his day.

III. The Use of Logarithms in Computation

147. To Find the Product of Several Numbers. The processes of multiplication, division, raising to powers, and extraction of roots, as carried out in arithmetic, may be greatly shortened by the use of logarithms, as we shall now show.

Let us take any two numbers, for example 25 and 37, and determine their logarithms. We find that log $25 = 1.3979$ and log $37 = 1.5682$. This means (§ 136) that

$$25 = 10^{1.3979} \quad \text{and} \quad 37 = 10^{1.5682}$$

Multiplying, we thus have

$$25 \times 37 = 10^{1.3979 + 1.5682} \qquad (\text{§ 117, Law I})$$

The last equality means (§ 136) that

$$\log (25 \times 37) = 1.3979 + 1.5682,$$

or

$$\log (25 \times 37) = \log 25 + \log 37.$$

Similarly, if we start with the *three* numbers 25, 37, and 18 we can show that

$$\log (25 \times 37 \times 18) = \log 25 + \log 37 + \log 18.$$

Thus we arrive at the following important rule.

Rule V. *The logarithm of a product is equal to the* ***sum*** *of the logarithms of its factors.*

Thus $\log (13 \times .0156 \times 99.8) = \log 13 + \log .0156 + \log 99.8.$

The way in which this rule is used to find the value of the product of several numbers is shown below.

Example 1. To find the value of $13 \times .0156 \times 99.8$

Solution.

$$\begin{aligned} \log\ 13 &= \ \ 1.1139 \\ \log\ .0156 &= \ \ 8.1931 - 10 \\ \log\ 99.8 &= \ \ \underline{1.9991 \qquad\ } \end{aligned}$$

Adding, $\quad 11.3061 - 10$, or 1.3061

Hence, by Rule V, the logarithm of the desired product is 1.3061. It follows that the product itself is the number whose logarithm is 1.3061. When we look up this number (as in § 146) we find it to be 20.23. Hence $13 \times .0156 \times 99.8 = 20.23$ (approximately). *Ans.*

Example 2. To find the value of

$$8.45 \times .678 \times .0015 \times 956 \times .111$$

Solution.

$$\begin{aligned}
\log\ 8.45 &= 0.9269\\
\log\ .678 &= 9.8312-10\\
\log\ .0015 &= 7.1761-10\\
\log\ 956 &= 2.9805\\
\log\ .111 &= \underline{9.0453-10}
\end{aligned}$$

Adding, $\quad 29.9600-30=9.9600-10.$

Hence, by Rule V, the logarithm of the desired product is seen to be $9.9600-10$.

Therefore the product itself is found (as in § 146) to be .912 (approximately). *Ans.*

These examples lead to the following rule.

Rule VI. *To multiply several numbers:*

1. *Add the logarithms of the several factors.*
2. *The sum thus obtained is the logarithm of the product.*
3. *The product itself can then be determined as in* § 146.

Note. It may happen (as in Example 2) that the sum of several logarithms is negative. In such cases it is best to write the sum in such a form that it will end with -10, thus conforming always to § 143.

EXERCISES

Find, by Rule V, § 147, the value of each of the following logarithms.

1. $\log (35.1 \times 7.29)$.
2. $\log (5 \times 3.17 \times .0016)$.
3. $\log (145.7 \times 8.35 \times .00456)$.
4. $\log (3.456 \times .001798 \times 1.456)$.

Find (by Rule VI, § 147) the value of

5. $56.8 \times 3.47 \times .735$

Check your answer by multiplying out the long way as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method. Compare also the time required for the two methods.

6. $.975 \times 42.8 \times 3.72$

7. $896 \times 40.8 \times 3.75 \times .00489$

8. $34.56 \times 18.16 \times .0157$

[HINT. See § 145.]

9. $576.8 \times 43.25 \times 3.576 \times .0576$

10. $60.573 \times 8.087 \times .008915 \times 1.2387$

11. $23 \times 23 \times 23 \times 23 \times 23 \times 23 \times 23$, (or 23^7)

12. $1.2 \times 2.3 \times 3.4 \times 4.5 \times 5.6 \times 6.7 \times 7.8$

13. $.31 \times 5.198 \times 6.831 \times 2.584 \times .00312 \times .07568$

14. Since $25 \times 15 = 375$ we know by Rule V, § 147, that the logarithm of 25 added to the logarithm of 15 is equal to the logarithm of 375. Show that the values given in the tables for log 25, log 15, and log 375 confirm this result. Invent and try out several other similar problems for yourself.

148. To Find the Quotient of Two Numbers. Let us take any two numbers, for example 41 and 29, and look up their logarithms. We find

$$\log\ 41 = 1.6128$$
$$\log\ 29 = 1.4624$$

These mean that

$$41 = 10^{1.6128}$$

and

$$29 = 10^{1.4624}$$

Whence, dividing the first of these equalities by the second, we obtain

$$41 \div 29 = \frac{10^{1.6128}}{10^{1.4624}} = 10^{1.6128 - 1.4624} \qquad (\S\ 117,\ \text{Law II})$$

The last equality means that

$$\log\ (41 \div 29) = 1.6128 - 1.4624 = \log\ 41 - \log\ 29.$$

This result illustrates the following general rule.

RULE VII. *The logarithm of a quotient is equal to the logarithm of the dividend* **minus** *the logarithm of the divisor.*

Thus log (467.3 ÷ .00149) = log 467.3 − log .00149

The way in which this rule is used to find the value of the quotient of two numbers is shown below.

EXAMPLE 1. To find the value of 236 ÷ 4.15

SOLUTION.

$$\begin{aligned} \log\ 236 &= 2.3729 \\ \log\ 4.15 &= \underline{0.6180} \end{aligned}$$

Subtracting, 1.7549

Hence the logarithm of the desired quotient is 1.7549 (Rule VII)

The number whose logarithm is 1.7549 is found (as in § 146) to be 56.875

Therefore 236 ÷ 4.15 = 56.875 (approximately). *Ans.*

EXAMPLE 2. To find the value of 1.46 ÷ .00576

SOLUTION.

$$\begin{aligned} \log 1.46 = 0.1644 &= 10.1644 - 10 \quad \text{(See Note below.)} \\ \log .00576 \quad &= \underline{\ \ 7.7619 - 10} \end{aligned}$$

Subtracting, 2.4025

The number whose logarithm is 2.4025 is found to be 252.64

Therefore 1.46 ÷ .00576 = 252.64 (approximately). *Ans.*

Thus we have the following rule.

RULE VIII. *To find the quotient of two numbers:*

1. *Subtract the logarithm of the divisor from the logarithm of the dividend.*

2. *The difference thus obtained is the logarithm of the quotient.*

3. *The quotient itself can then be determined as in* § 146.

NOTE. To subtract a negative logarithm from a positive one, or to subtract a greater logarithm from a less, increase the characteristic of the minuend by 10, writing −10 after the mantissa to compensate. Thus, in Example 2, we wished to subtract the negative logarithm 7.7619 − 10 from the positive one 0.1644. Therefore 0.1644 was written in the form 10.1644 − 10, after which the subtraction was easily performed.

EXERCISES

Find, by Rule VII, § 148, the value of each of the following logarithms.

1. log $(13 \div 9)$.
2. log $(217 \div 8.16)$.
3. log $(38.76 \div .0017)$.
4. log $(8.764 \div 114.3)$.

Find, by Rule VIII, § 148, the value of each of the following quotients.

5. $246 \div 15.7$

Check your answer by dividing out the long way as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

6. $34.7 \div 5.34$
7. $389.7 \div 4.353$

[Hint. See § 145.]

8. $45.67 \div 38.01$
9. $3.25 \div .00876$

[Hint. See Note in § 148.]

10. $49.6 \div 87.3$
11. $\dfrac{40.3 \times 6.35}{3.72}$

[Hint. Find the logarithm of the numerator by Rule V, § 147.]

12. $\dfrac{.0036 \times 2.36}{.0084}$
13. $\dfrac{24.3 \times .695 \times .0831}{8.40 \times .216}$

14. Since $27 \div 9 = 3$ we know, by Rule VII, § 147, that the logarithm of 9 subtracted from the logarithm of 27 is equal to the logarithm of 3. Show that the values given in the tables for log 9, log 27, and log 3 confirm this result. Invent and try out several other similar problems for yourself.

149. To Raise a Number to a Power. Let us take any number, for example 25, and raise it to any power, say the fourth. We then have 25^4, which means $25 \times 25 \times 25 \times 25$.

Hence, by Rule V, § 147, we must have

$\log 25^4 = \log 25 + \log 25 + \log 25 + \log 25$, or $\log 25^4 = 4 \log 25$.

This illustrates the following rule.

RULE IX. *The logarithm of any power of a number is equal to the logarithm of the number* **multiplied by** *the exponent indicating the power.*

Thus $\log 3.17^{10} = 10 \log 3.17$; similarly, $\log .00174^6 = 6 \log .00174$.

The way in which this principle is used to raise a number to a power is shown below.

EXAMPLE 1. To find the value of 2.37^4

SOLUTION. $\log 2.37 = 0.3747$

$$\begin{array}{r} 0.3747 \\ 4 \\ \hline 1.4988 \end{array}$$

Multiplying, 1.4988

Hence $\log 2.37^4 = 1.4988$ (Rule IX)

The number whose logarithm is 1.4988 is found to be 31.535

Therefore

$2.37^4 = 31.525$ (approximately). *Ans.*

EXAMPLE 2. To find the value of $.856^5$

SOLUTION. $\log .856 = 9.9325 - 10$

$$\begin{array}{r} 9.9325-10 \\ 5 \\ \hline 49.6625-50 \end{array}$$

Multiplying, $49.6625 - 50 = 9.6625 - 10$

The number whose logarithm is $9.6625 - 10$ is .4597 (§ 146)

Therefore

$.856^5 = .4597$ (approximately). *Ans.*

Thus we have the following rule.

RULE X. *To raise a number to a power:*

1. *Multiply the logarithm of the number by the exponent indicating the power.*
2. *The result thus obtained is the logarithm of the answer.*
3. *The answer itself can then be determined as in* § 146.

EXERCISES

Find, by Rule IX, § 149, the value of each of the following logarithms.

1. $\log 16^5$ 2. $\log 3.12^3$ 3. $\log .0176^2$ 4. $\log 36.64^4$

Find, by Rule X, § 149, the value of each of the following expressions.

5. 8.82^3

Check your answer by raising 8.82 to the third power as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

6. 4.12^4 **7.** 4.123^4

8. $.175^5$ [HINT. See Ex. 2 in § 149.]

9. $81^3 \times .015^2$ [HINT. Combine the rules of §§ 147 and 149.]

10. $43 \times 8.9^2 \times .075^3$

11. $\dfrac{8.76 \times 53.9 \times 4.5^3}{2.3^2 \times 3.15 \times 5.14^3}$

[HINT. Use Rules VI, VIII, X.]

12. Since $9^3 = 729$ we know, by Rule IX, § 149, that three times the logarithm of 9 is equal to the logarithm of 729. Show that the values given in the tables for log 9 and log 729 confirm this result. Invent and try out several other similar problems for yourself.

150. To Extract Any Root of a Number. Let us take any number, for example 36, and consider any root of it, say the fifth, that is, let us consider $\sqrt[5]{36}$.

Supposing x to be the value of the desired root, we have

$$x^5 = 36. \qquad (\S\ 118)$$

Now the logarithm of the first member of this equality is equal to 5 log x by Rule IX.

Hence $5 \log x = \log 36$, or $\log x = \frac{1}{5} \log 36$.

This illustrates the following rule.

RULE XI. *The logarithm of the root of a number is equal to the logarithm of the radicand* ***divided by*** *the index of the root.*

Thus $\log \sqrt[4]{2.73} = \frac{1}{4} \log 2.73$; similarly, $\log \sqrt[7]{.01685} = \frac{1}{7} \log .01685$.

The way in which this principle is used to extract the roots of numbers in arithmetic will now be shown.

EXAMPLE 1. To find the value of $\sqrt[4]{85.2}$

SOLUTION. $\log 85.2 = 1.9304$,

so that $\frac{1}{4}$ of $\log 85.2 = 0.4826$.

Hence $\log \sqrt[4]{85.2} = 0.4826$. (Rule XI)

The number whose logarithm is 0.4826 is 3.038 (§ 146)

Therefore $\sqrt[4]{85.2} = 3.038$ (approximately). *Ans.*

EXAMPLE 2. To find the value of $\sqrt[5]{.0875}$

SOLUTION. $\log .0875 = 8.9420 - 10$,

so that $\frac{1}{5}$ of $\log .0875 = \frac{1}{5}(8.9420 - 10) = \frac{1}{5}(48.9420 - 50)$

$= 9.7884 - 10$. (See Note below.)

The number whose logarithm is $9.7884 - 10$ is .6143 (§ 146)

Therefore $\sqrt[5]{.0875} = .6143$ (approximately). *Ans.*

These examples lead to the following rule.

RULE XII. *To find any root of any number.*

1. *Divide the logarithm of the number by the index of the root.*

2. *The quotient thus obtained is the logarithm of the desired root.*

3. *The root itself can then be determined as in* § 146.

NOTE. To divide a negative logarithm, write it in a form where the negative part of the characteristic may be divided exactly by the divisor giving -10 as quotient. Thus, in Example 2, we wrote $8.9420 - 10$ in the form $48.9420 - 50$ after which the division by 5 was easily done and resulted in a form ending in -10.

EXERCISES

Find, by Rule XI, § 150, the value of each of the following logarithms.

1. $\log \sqrt[5]{16}$ 2. $\log \sqrt[3]{3.12}$ 3. $\log \sqrt[4]{.0175}$ 4. $\log \sqrt[5]{38.56}$

Find, by Rule XII, § 150, the value of each of the following.

5. $\sqrt{315}$

Check your answer by extracting the square root of 315 (correct to three decimal places) as in arithmetic. Compare the two results and see how great was the error committed by following the short (logarithmic) method.

6. $\sqrt[3]{4.32}$

7. $\sqrt[3]{4.325}$

8. $\sqrt[5]{.0957}$

[Hint. See Example 2 in § 150.]

9. $\sqrt[4]{8.76 \times .0153}$

[Hint. Use Rules IX and XI.]

10. $\sqrt[3]{576} \times \sqrt[4]{8.76}$

11. $\sqrt{\dfrac{576 \times 9.13^2}{3.8 \times 5.32^3}}$.

12. Since $\sqrt{49}=7$ we know, by Rule XI, § 150, that one half the logarithm of 49 is equal to the logarithm of 7. Show that the values given in the tables for log 49 and log 7 confirm this result. Invent and try out several other similar problems for yourself.

APPLIED PROBLEMS

Solve the following exercises by logarithms.

1. How many cubic feet of air are there in a schoolroom whose dimensions are 50.5 ft. by 25.3 ft. by 10.4 ft.?

2. How many gallons will a rectangular tank hold whose dimensions are 8 ft. 10 in. by 9 ft. 3 in. by 10 ft 1 in.?

3. How much wheat will a cylindrical bin hold if the diameter of the base is 9 ft. 5 in. and the height is 40 ft. 4 in.?

4. How much would a sphere of solid cork weigh if its diameter was 4 ft. 3 in., it being known that the specific gravity of cork is .24? (See Example 14 (*e*), page 6.)

[Hint. To say that the specific gravity of cork is .24 means that any volume of cork weighs .24 times as much as an *equal* volume of water. Water weighs 62.5 pounds per cubic foot.]

5. The diameter d in inches of a wrought-iron shaft required to transmit h horse power at a speed of n revolutions per minute is given by the formula $d=\sqrt[3]{\dfrac{65\,h}{n}}$. Find the diameter required when 135 horse power is to be transmitted at a speed of 130 revolutions per minute.

6. The amount to which P dollars will accumulate at $r\%$ compound interest in n years is given by the formula

$$A=P\left(1+\frac{r}{100}\right)^n.$$

Find A if $P=\$500$, $r=5$, and $n=10$.

Find A if $P=\$100$, $r=3.5$, and $n=15$.

7. By means of Formula 3 of § 65, find the area of the triangle whose sides are 3.15 in., 4.87 in., and 2.68 in.

8. The height H of a mountain in feet is given by the formula

$$H=49{,}000\left(\frac{R-r}{R+r}\right)\left(1+\frac{T+t}{900}\right),$$

where R, r are the observed heights of the barometer in inches at the foot and at the summit of the mountain, and where T, t are the observed Fahrenheit temperatures at the foot and summit.

Find the height of a mountain if the height of the barometer at the foot is 29.6 inches and at the summit 25.35 inches, while the temperature at the foot is 67° and at the summit 32°.

9. By means of the formula in Ex. 6 answer the following question: How long will it take a sum of money to double itself if placed at compound interest at 5%?

14.2 years. *Ans.*

General Logarithms

151. Logarithms to Any Base.** In § 136 we defined the logarithm of a number as the power to which 10 must be raised to obtain that number. Thus, from such equalities as $10^2=100$, $10^3=1000$, etc., we had $\log 100=2$, $\log 1000=3$, etc. Strictly speaking, this defines the logarithm of a number *to the base 10*, or, as it is usually called, a ***common logarithm.

We may and frequently do use some other base than 10. For example, since $3^2=9$, $3^3=27$, $3^4=81$, etc., we can say that the logarithm of 9 *to the base 3* is 2, the logarithm of 27 *to the base 3* is 3,

the logarithm of 81 *to the base 3* is 4, etc. The usual way of denoting this is to write $\log_3 9 = 2$, $\log_3 27 = 3$, $\log_3 81 = 4$, etc. Observe that the number being used as the base is thus placed to the right and just below the symbol log.

Similarly, we have $\log_2 16 = 4$, $\log_8 64 = 2$, $\log_5 125 = 3$, etc.

Thus we have the following general definition. *The logarithm of any number x to a given base a is the power of a required to give x. It is written* $\log_a x$. Any positive number except 1 may be used as the base.

NOTE. When the base a is taken equal to 10 (that is, in the usual case) we write simply $\log x$ instead of $\log_{10} x$.

EXERCISES

State first the *meaning* and then the *value* of

1. $\log_2 4$. **2.** $\log_2 8$. **3.** $\log_4 16$. **4.** $\log_3 \frac{1}{3}$.
5. $\log_2 \frac{1}{4}$. **6.** $\log_4 \frac{1}{64}$. **7.** $\log_5 .2$ **8.** $\log_8 32$.

***152. Logarithm of a Product.** We can now show that Rule V, § 147, holds true *whatever the base.* That is, if M and N are any two numbers, and a the base, then

$$\log_a MN = \log_a M + \log_a N.$$

PROOF. Let $x = \log_a M$ and $y = \log_a N$. Then $a^x = M$ and $a^y = N$ (§ 151). Hence $a^x \cdot a^y = MN$, or $a^{x+y} = MN$ (§ 117, Law I). But the last equality means that

$$\log_a MN = x + y = \log_a M + \log_a N. \qquad (§\ 151)$$

***153. Logarithm of a Quotient.** Rule VII, § 148, holds true *whatever the base.* That is, if M and N are any two numbers, then

$$\log_a (M \div N) = \log_a M - \log_a N.$$

PROOF. Let $x = \log_a M$ and $y = \log_a N$. Then $a^x = M$ and $a^y = N$. (§ 151). Hence, $a^x \div a^y = M \div N$, or $a^{x-y} = M \div N$ (§ 117, Law II). But the last equality means that

$$\log_a (M \div N) = x - y = \log_a M - \log_a N. \qquad (§\ 151)$$

***154. Logarithm of a Power of a Number.** Rule IX, § 149, holds true *whatever the base.* That is, if M is any number and n any (positive integral) power, then

$$\log_a M^n = n \log_a M.$$

PROOF. Let $x = \log_a M$. Then $a^x = M$ (§ 151) and hence $a^{nx} = M^n$ (§ 117, Law III). But the last equality means that

$$\log_a M^n = nx = n \log_a M. \qquad (§\ 151)$$

***155. Logarithm of a Root of a Number.** Rule XI, § 150, holds true *whatever the base.* That is, if M is any number and n any (positive integral) root, then

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M.$$

PROOF. Let $x = \log_a M$ Then $a^x = M$ (§ 151) and hence $(a^x)^{1/n} = M^{1/n}$, or $a^{x/n} = \sqrt[n]{M}$ (§ 121). But the last equality means that

$$\log_a \sqrt[n]{M} = \frac{x}{n} = \frac{1}{n} \log_a M.$$

***156. Summary.** From the results established in §§ 151–155 it appears that Rules V–XII, §§ 147–150, are not only true when the base is 10 (as was there taken) but they are true for *any* base. Complete tables have been worked out for various bases other than 10, but we shall not consider them further here.

NOTE. The reason why 1 cannot be used as a base is that 1 to *any* power is equal to 1, that is, we cannot get different numbers by raising 1 to different powers.

***157. Historical Note.** Logarithms were first introduced and employed for shortening computation by JOHN NAPIER (1550–1617), a Scotchman. (See the picture facing p. 233.) However, he did not use the base 10, this being first done by the English mathematician BRIGGS (1556–1631), who computed the first table of *common* logarithms and did much to bring logarithms into general use.

158. Calculating Machines. The Slide-Rule.** Machines have been invented and are now coming into very general use, especially by engineers, by which the processes of multiplication, division, involution, and evolution can be immediately performed. The construction of these machines depends upon the principles of logarithms, but to describe the machines and their methods of working would take us beyond the scope of this text. The simplest machine of this kind is the ***slide rule, the use of which is easily understood. A simple slide rule with directions is inexpensive and may ordinarily be secured from booksellers.

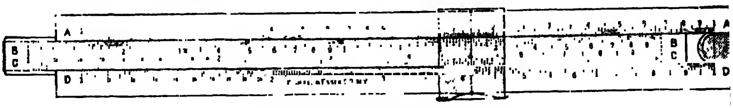


FIG. 70. THE SLIDE RULE.

PART III. SUPPLEMENTARY TOPICS

CHAPTER XXI

FUNCTIONS

159. The Function Idea. In ordinary speech we make such statements as the following:

1. The area of a circle depends upon the length of its radius.

2. The time it takes to go from one place to another depends upon the distance between them.

3. The power which an engine can exert depends upon the pressure per square inch of the steam in the boiler.

Another way of stating these facts is as follows:

1. The area of a circle is a *function* of the length of its radius.

2. The time it takes to go from one place to another is a *function* of the distance between them.

3. The power which an engine can exert is a *function* of the pressure per square inch of the steam in the boiler.

The idea thus conveyed by the word ***function*** *is that we have one magnitude whose value is determined as soon as we know the value of some other one (or more) magnitudes upon which the first one depends.* This idea is at once seen to be universal in everyday experience and for that reason it becomes of great importance in mathematics.† In the present

† The extended formal study of the function idea enters into that branch of mathematics known as the *Calculus*.

chapter we shall indicate briefly how it is related to some of the subjects treated in the preceding chapters, noting especially the significance of the idea when considered graphically.

160. Types of Algebraic Functions. An expression of the form

(1) $$a_0x + a_1,$$

where the coefficients a_0 and a_1 have any given values (except a_0 must not be 0) is called a ***linear function*** of x. Observe that every such expression depends for its value upon the value assigned to x, and is determined as soon as x is known. Hence it is a function of x in the sense explained in § 159. It is called a *linear* function since it is of the first degree in x. (Compare § 26.)

For example, $2x+3$ is a linear function of x. Here we have the form (1) in which $a_0=2$ and $a_1=3$. Similarly, $3x-2$, $x-4$, $-x+\frac{1}{4}$ and $3x$ are linear functions of x. (Why?)

Likewise, $3t+2$ is a linear function of t, while $-r+5$ is a linear function of r, etc.

As an example of a linear function in everyday experience, suppose that in Fig. 71 a person starts from the point P and moves to the right at the rate of 15 miles per hour, and let Q be the point 10

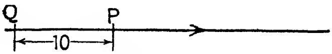


FIG. 71.

miles to the left of P. Then we may say that the distance of the traveler from Q is a linear function of the time he has been traveling, for if t represent the number of hours he has been traveling, his distance from P is $15t$ (see § 62) and hence his distance from Q is $15t+10$. This is seen to be a linear function of t, being of the form (1) in which $a_0=15$ and $a_1=10$.

Likewise, the interest which a given principal, P, will yield in one year is a linear function of the rate, for, if r be the rate, the interest

in question is given by the formula $P\times\frac{r}{100}$, or $\frac{P}{100}r$, and this is seen to be of the form (1) in which $a_0=\frac{P}{100}$, and $a_1=0$.

An expression of the form

$$(2) \qquad a_0x^2+a_1x+a_2,$$

where the coefficients a_0, a_1, and a_2 have any given values (except that a_0 must not be 0) is called a ***quadratic function*** of x.

For example, $2x^2+3x-1$ is a quadratic function of x because it is of the form (2) in which $a_0=2$, $a_1=3$, $a_2=-1$. Likewise, $x^2+\frac{1}{4}x$; $x^2+\frac{1}{4}$; $-x^2+3x$; $5x^2$; x^2 are quadratic functions of x. (Why?)

Again, we may say that the area of a square is a quadratic function of the length of one side, for if x be the length of side, the area is x^2 and this is of the form (2) in which $a_0=1$, $a_1=a_2=0$.

Similarly, the area of a circle is a quadratic function of the radius. (Why?)

An expression of the form

$$(3) \qquad a_0x^3+a_1x^2+a_2x+a_3,$$

where the coefficients a_0, a_1, a_2 and a_3 have any given values (except that a_0 must not be 0) is called a ***cubic function*** of x.

For example, $3x^3-x^2+\frac{1}{2}x-1$; $4x^3-x$; x^3-2x^2+1; $5x^3$; x^3, etc. (Why?)

Again, we may say that the volume of a cube is a cubic function of the length of one edge. (Why?) Also, the volume of a sphere is a cubic function of the radius. (Why?)

It may now be observed that the expressions (1), (2), and (3) are but special forms of the more general expression

$$(4) \qquad a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$$

where it is understood that n can be any positive integer, while the coefficients a_0, a_1, a_2, $\cdots$ a_n have any given values (except that a_0 must not be 0). This is called the ***general***

integral rational function of x, or, more simply, a ***polynomial*** in x. It reduces to the linear function (1) when $n=1$; to the quadratic function (2) when $n=2$; etc.

Expressions such as

$$\sqrt{x},\ \sqrt[3]{x},\ \sqrt[4]{x},\ 3\sqrt{x}+\sqrt[5]{x},\ x^2+4\,x^{-\frac{2}{3}},\quad \frac{5\,x}{\sqrt[3]{x}-1}$$

and all others composed merely of powers or roots (or both) of x are classed under the name of ***algebraic functions***. Since all functions of the form (4) are composed of integral powers only of x, they are but special cases of the algebraic functions just mentioned.

EXERCISES

1. Show that the thickness of a book is a linear function of the number of its pages.

[HINT. Let x be the number of pages, d be the thickness of each page, and D the thickness of each cover. Now build up the formula for the thickness of the book and note which of the functional types in § 160 is present.]

2. The supply of gasoline in a tank was very low, its depth being but 1 inch all over the bottom, when it was replenished from a pipe which delivered 3 gallons per minute. Show that the amount in the tank at any moment during the filling was a linear function of the time since the filling began.

3. Show that the force which a steam engine has at any moment at its cylinder is a linear function of the area of the piston; also that it is a linear function of the boiler pressure of the steam per square inch.

4. A certain room contains a number of 16-candle-power electric lights and a number of Welsbach gas-burners. Show that the amount of illumination at any time is a linear function of the number of electric lights turned on. Is this true regardless of the number of gas-burners already lighted?

5. Show that the perimeter of a square is a linear function of the length of one side; also that the circumference of a circle is a linear function of its radius.

6. Show that if each side of a square be increased by x, the corresponding increase in the area will be a quadratic function of x.

[HINT. Let a = the length of one side of the original square. Then the area is a^2 and the area of the new square is $(a+x)^2$. Now formulate the expression for the *increase* in area.]

7. Show that if the radius of a circle be increased by x, the corresponding increase in area will be a quadratic function of x.

8. Show that if the edge of a cube be increased by x the corresponding increase in volume will be a cubic function of x. State and prove the corresponding statement for a sphere.

9. Show that if y varies directly as x (see § 113), then y is a linear function of x. Is the *converse* of this statement necessarily true, namely if y is a linear function of x, then y varies directly as x?

10. When y varies as the square of x, to which one of the functional types mentioned in § 160 does y belong? Answer the same question when y varies inversely as x; when y varies inversely as the square of x.

11. A certain linear function of x takes the value 5 when $x=1$ and takes the value 8 when $x=2$. Determine completely the form of the function.

SOLUTION. Since the function is linear, it is of the form a_0x+a_1. Since this expression must (by hypothesis) be equal to 5 when $x=1$, we have $a_0 \cdot 1+a_1=5$. Likewise, placing $x=2$, gives $a_0 \cdot 2+a_1=8$. Solving these two equations for a_0 and a_1 we obtain $a_0=3$, $a_1=2$. The desired function is therefore $3x+2$. *Ans.*

12. A certain linear function of x takes the value 14 when $x=3$, and takes the value -6 when $x=-1$. Determine completely the form of the function.

13. A certain quadratic function takes the value 0 when $x=1$, and the value 1 when $x=2$, and the value 4 when $x=3$. Determine completely the form of the function.

14. Show that the area of any triangle is an algebraic function of the sum of its three sides. (See Formula 3 in § 65.)

161. Functions Considered Graphically. By the ***graph of a function*** is meant the line or curve which results when some letter, as y, is placed equal to the function and the graph is drawn of the equation thus obtained. The purpose of the graph is to bring out clearly and quickly to the eye the relation between the given function and the quantity (variable) upon which it depends for its values.

The method of drawing such graphs is precisely the same as that given in § 29, p. 43 for equations of the first degree, and in § 57, p. 90, for quadratic equations.

Thus, in order to obtain the graph of the function x^3, we place $y=x^3$ and proceed to draw the graph of this equation in the way explained in § 29, that is, we assign various values to x and compute (from this equation) the corresponding values of y, then we plot each point thus obtained and finally draw the smooth curve passing through all such points.

Below is a table of several values of x and y thus computed; and the graph is shown in Fig. 72.

When $x=$	-2	-1	0	1	2	3	4
then $y=$	-8	-1	0	1	8	27	64

The portion of the curve lying to the right of the y-axis extends upward indefinitely, while the portion to the left of the same axis extends downward indefinitely. Note that, from the way this curve has

been drawn, it at once brings out to the eye the value of the given function x^3 for any value of the letter x upon which this function depends, the function values being the *ordinates* (§ 28) of the points on the curve. For example, at $x=2$ the corresponding ordinate measures 8, which is the function value then present.

This curve may be used as a *graphical table of cubes* of numbers. Thus, if $x=1.5$, $y=3.4$, approximately, etc. Likewise, if y is given first, the curve shows the *cube root* of y; for example, if $y=4$, x is about 1.6. The figure may be drawn by the student on a much larger scale; the values of x and y can be read much more accurately from such a figure than from the small figure on this page.

Another means of improving the accuracy of the figure is to take a longer distance on the horizontal line to represent one unit than is taken to represent one unit on the vertical scale.

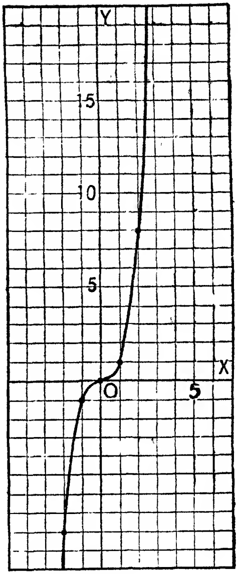


FIG. 72.

The graph of every linear function is a straight line. The graph of every other algebraic function is a curved line.

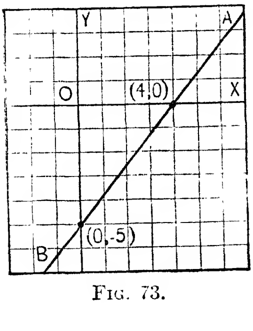


FIG. 73.

For example, in considering the graph of the linear function $\frac{5}{4}x-5$, we place $y=\frac{5}{4}x-5$. But this is an equation of the first degree between x and y and hence (§ 29) its graph is a straight line. Fig. 73 shows the result.

Note that the graph cuts the x-axis in *one* point. The abscissa of this particular point is 4, which indicates that 4 is the root, or solution, of the equation $\frac{5}{4}x-5=0$, for it is this value of x that makes $y=0$.

The graph of every quadratic function belongs to the class of curves known as *parabolas*. A parabola resembles in form an oval, open at one end. It never cuts the x-axis in more than *two* points.

Fig. 74 shows the graph of the quadratic function x^2+x-2. Note that the curve cuts the x-axis at two points whose abscissas are -2 and 1, respectively. This indicates that -2 and 1 are the roots of the quadratic equation $x^2+x-2=0$.

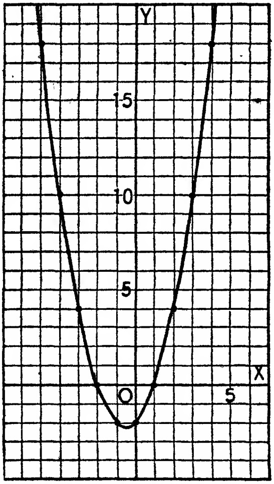


Fig. 74.

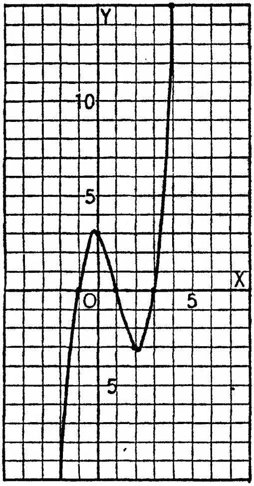


Fig. 75.

The general form of the graph of a cubic function is that of an indefinitely long smooth curve which cuts the x-axis in no more than *three* points.

Fig. 75 shows the graph of the cubic function x^3-3x^2-x+3. It cuts the x-axis at three points whose abscissas are respectively -1, 1, and 3. These values, therefore, are the roots of the cubic equation $x^3-3x^2-x+3=0$.

Similarly, the general form of the graph of the rational integral function of the fourth degree is that of an indefinitely long smooth curve which cuts the x-axis in no more than *four* points. And it may be said likewise that the graph of the general integral function of degree n (see (4), § 160) is an indefinitely long smooth curve which cuts the x-axis in no more than n points.

Fig. 76 shows, for example, the graph of $2x^4-5x^3+5x-2$, this being a function of the fourth degree. The four points where the curve cuts the x-axis have abscissas which are equal respectively to -1, $\frac{1}{2}$, 1, and 2. These values, therefore, are the roots of the equation $2x^4-5x^3+5x-2=0$.

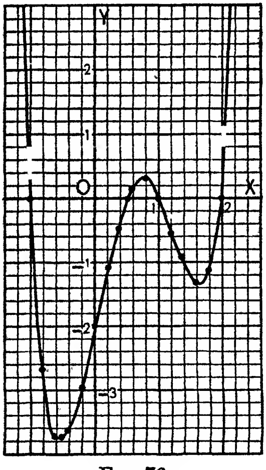


FIG. 76.

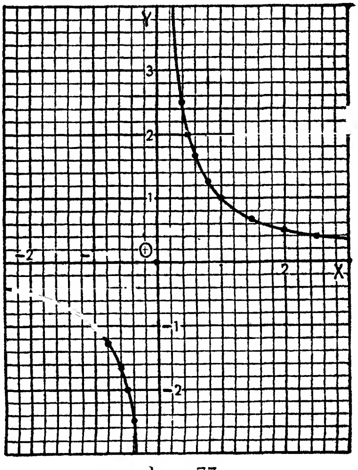


FIG. 77.

Fractional expressions give rise to more complex graphs, which may have more than one piece. Fig. 77 shows, for example, the graph of $1/x$. If we let $y=1/x$, *y varies inversely as x* (§ 110). The curve is therefore similar to those drawn in § 115, Fig. 69. The graph consists of two branches and belongs to the class of curves known as *hyperbolas*. These we have already met in § 78.

EXERCISES

Draw the graphs of the following functions by plotting several points on each and drawing the curve through them. Try to plot enough points so that the form and location of the various waves, or arches, of the curve will be brought out clearly, as in the figures of § 161. Note how many times the curve cuts the x-axis and make such inferences as you can regarding the roots of the corresponding equation.

[HINT. When the graph of a quadratic function fails to cut the x-axis, this indicates that the roots of the corresponding quadratic equation are imaginary. (See §§ 57, 60.) Similarly, when the graph of a cubic function cuts the x-axis in but one point, this indicates that there is but one real root to the corresponding equation, the other two roots being imaginary. In general, the number of times the graph cuts the x-axis indicates the number of *real* roots of the corresponding equation, the number of imaginary roots being the degree of the equation minus the number of real roots.]

1. $3x+4$. **2.** x. **3.** x^2-x-2. **4.** x^2-4.

5. x^2+1. **6.** x^3-3x^2-x+3. **7.** x^3+3x^2+2x+6.

CHAPTER XXII

MATHEMATICAL INDUCTION — BINOMIAL THEOREM

162. Mathematical Induction. The three following purely arithmetic relations are easily seen to be true:

$$1+2=\tfrac{2}{2}(2+1),$$
$$1+2+3=\tfrac{3}{2}(3+1),$$
$$1+2+3+4=\tfrac{4}{2}(4+1).$$

We might at once infer from these that if n be *any* positive integer, there exists the algebraic relation

$$1+2+3+4+\cdots+n=\frac{n}{2}(n+1), \tag{1}$$

the dots indicating that the addition of the terms on the right continues up to and including the number n.

For example, if $n=8$, this would mean that

$$1+2+3+4+5+6+7+8=\tfrac{8}{2}(8+1).$$

Again, if $n=10$, it would mean that

$$1+2+3+4+5+6+7+8+9+10=\tfrac{10}{2}(10+1).$$

That these are indeed true relations is discovered as soon as we simplify them. Let the pupil convince himself on this point.

It is now to be carefully observed that the inference just made, namely that (1) is true for *any* n, is not yet justified, strictly speaking, from anything we have done, for we have only shown that (1) holds good for certain *special* values of n, and we could never hope to do more than this however long we continued to try out the formula in this way.

Something more than a knowledge of special cases must always be known before any perfectly certain *general* inference can be made. For example, the fact that Saturday was cloudy for 38 weeks in succession gives no certain information that it will be so on the 39th week.

We shall now show how the general formula (1) may be established free from all objection, that is in a way that leaves no possible question as to its truth in all cases.

Let r represent any one of the *special* values of n for which we know (1) to be true. Then

$$(2) \qquad 1+2+3+4+\cdots+r=\frac{r}{2}(r+1).$$

Let us add $(r+1)$ to both sides. The result is

$$1+2+3+4+\cdots+r+(r+1)=\frac{r}{2}(r+1)+(r+1).$$

In the second member of the last equation we may write

$$\frac{r}{2}(r+1)+(r+1)=(r+1)\left(\frac{r}{2}+1\right)=(r+1)\left(\frac{r+2}{2}\right)=\frac{r+1}{2}(r+2),$$

while the first member has the same meaning as

$$1+2+3+\cdots+(r+1).$$

Thus, (2) being given us, it follows that we may write

$$(3) \qquad 1+2+3+4+\cdots+(r+1)=\frac{r+1}{2}(r+2).$$

But (3) is seen to be precisely the same as (2) except that $r+1$ now replaces r throughout. Stated in words, this result means that if (1) is true when $n=r$, as we have supposed, then it holds true *necessarily* for the next greater value of n, which is $r+1$.

The original fact which we wished to establish (namely, that (1) is true for *any* n) now follows without difficulty. In fact, we know (see beginning of this section) that (1) is true when $n=4$, from which it now follows that it must be true also when $n=5$. Being true when $n=5$, the same reasoning says it must be true also when $n=6$. Being true when $n=6$, it must be likewise true when $n=7$. Proceeding in this way, we may reach any integer n we may mention, however large it may be. Hence (1) is true for any such value of n.

This method of reasoning illustrates what is termed ***mathematical induction***. Another example of the process will now be given, the steps being arranged, however, in a more condensed form.

EXAMPLE. Prove by mathematical induction that

(1) $1+3+5+7+\cdots+(2n-1)=n^2$. ($n$ = *any positive integer*)

SOLUTION. When $n=1$, the formula gives $1=1^2$; when $n=2$, it gives $1+3=2^2$; when $n=3$, it gives $1+3+5=3^2$, all of which arithmetical relations are seen to be correct.

Let r represent any value of n for which the formula has been proved. Then

(2) $$1+3+5+7+\cdots+(2r-1)=r^2.$$

Adding $(2r+1)$ to each member gives

(3) $$1+3+5+7+\cdots+(2r+1)=r^2+(2r+1)=r^2+2r+1=(r+1)^2.$$

But (3) is the same as (2) except that r has been replaced throughout by $r+1$. Hence, if (1) is true for any value of n, such as r, it is necessarily true also for that value of n increased by 1.

Now, we know (1) to be true when $n=3$. (See above.) Hence it must be true when $n=4$. Being true when $n=4$, it must be true when $n=5$, etc., and in this way we now know that (1) is true for any value (positive integral) of n whatever.

EXERCISES

Prove the correctness of each of the following formulas by mathematical induction, n always being understood to be *any* positive integer.

1. $2+4+6+8+\cdots+2n=n(n+1)$.

[HINT. First try out for $n=1$, $n=2$, and $n=3$. Let r represent a number for which the formula holds. Add $2(r+1)$ to both members of the resulting equation and compare results.]

2. $3+6+9+12+\cdots+3n=\frac{3n}{2}(n+1)$.

3. $1^2+2^2+3^2+4^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

4. $2^2+4^2+6^2+\cdots+(2n)^2=\frac{2}{3}n(n+1)(2n+1)$.

5. $1^3+2^3+3^3+4^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$.

6. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

7. $2+2^2+2^3+2^4+\cdots+2^n=2(2^n-1)$.

8. Prove that if n is any positive integer, a^n-b^n is divisible by $a-b$.

[HINT. Since $a^{r+1}-b^{r+1}=a(a^r-b^r)+b^r(a-b)$, it follows that $a^{r+1}-b^{r+1}$ will be divisible by $a-b$ whenever a^r-b^r is divisible by $a-b$.]

9. Prove that $a^{2n}-b^{2n}$ is divisible by $a+b$.

163. The Binomial Theorem. If we raise the binomial $(a+x)$ to the second power, that is find $(a+x)^2$, the result is $a^2+2\,ax+x^2$ (§ 10). Similarly, by repeated multiplication of $(a+x)$ into itself, we can find the expanded forms for $(a+x)^3$, $(a+x)^4$, $(a+x)^5$, etc. The results which we find in this way have been placed for reference in a table below:

$$(a+x)^2=a^2+2\,ax+x^2.$$
$$(a+x)^3=a^3+3\,a^2x+3\,ax^2+x^3.$$
$$(a+x)^4=a^4+4\,a^3x+6\,a^2x^2+4\,ax^3+x^4.$$
$$(a+x)^5=a^5+6\,a^4x+10\,a^3x^2+10\,a^2x^3+6\,ax^4+x^5,\text{ etc.}$$

Upon comparing these, we see that the expansion of $(a+x)^n$, *where n is any positive integer*, has the following properties:

1. *The exponent of a in the first term is n, and it decreases by 1 in each succeeding term.*

The last term, or x^n, may be regarded as a^0x^n. (See § 122.)

2. *The first term does not contain x. The exponent of x in the second term is 1 and it increases by 1 in each succeeding term until it becomes n in the last term.*

3. *The coefficient of the first term is 1; that of the second term is n.*

4. *If the coefficient of any term be multiplied by the exponent of a in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.*

For example, the term $6\,a^2x^2$, which is the *third* term in the expansion of $(a+x)^4$ (see p. 258) has a coefficient, namely 6, which may be derived by multiplying the coefficient of the preceding term (which is 4) by the exponent of a in that term (which is 3) and dividing the product thus obtained by the number of that term (which is 2).

5. *The total number of terms in the expansion is* $n+1$.

The results just observed regarding the expansion of $(a+x)^n$, where n is any positive integer, may be summarized and condensed into a single formula as follows:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots,$$

the dots indicating that the terms are to be supplied in the manner indicated up to the last one, or $(n+1)$st.

This formula is called the ***binomial theorem.*** By means of it, one may write down at once the expansion of any binomial raised to any positive integral power. That the formula is true in all cases, when n is a positive integer, will be proved in detail in § 165. We assume its truth here for those small values of n for which its correctness is easily tested.

NOTE. The formula is generally attributed to *Sir Isaac Newton* (1642–1727); see the picture facing p. 193.

EXAMPLE 1. Expand $(a+x)^6$.

SOLUTION. Here $n=6$, so the formula gives

$$(a+x)^6 = a^6 + 6\,a^5x + \frac{6\cdot 5}{1\cdot 2}a^4x^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3x^3 + \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}a^2x^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}ax^5 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6.$$

Simplifying the various coefficients by performing the possible cancelations in each, we obtain

$(a+x)^6 = a^6 + 6\,a^5x + 15\,a^4x^2 + 20\,a^3x^3 + 15\,a^2x^4 + 6\,ax^5 + x^6$. *Ans.*

NOTE. It may be observed that the coefficients of the first and last terms turn out to be the same; likewise the coefficients of the second and next to the last terms are the same, and so on symmetrically as we read the expansion from its two ends. This feature is true of the expansion of $(a+x)$ to *any* power. (Note the expansions of $(a+x)^2$, $(a+x)^3$, $(a+x)^4$, etc., as given at the beginning of § 163.)

EXAMPLE 2. Expand $(2-m)^5$.

SOLUTION. Here $a=2$, $x=-m$, and $n=5$. The formula thus gives

$$(2-m)^5=2^5+5\cdot 2^4(-m)+\frac{5\cdot 4}{1\cdot 2}\cdot 2^3(-m)^2+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\cdot 2^2(-m)^3$$
$$+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}\cdot 2(-m)^4+\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}(-m)^5.$$

Simplifying the coefficients (as in Example 1) this becomes

$$(2-m)^5=2^5+5\cdot 2^4(-m)+10\cdot 2^3(-m)^2+10\cdot 2^2(-m)^3$$
$$+5\cdot 2(-m)^4+(-m)^5.$$

Making further simplifications, we obtain

$$(2-m)^5=32-80\,m+80\,m^2-40\,m^3+10\,m^4-m^5. \quad \textit{Ans.}$$

NOTE. The result for $(2-x)^5$ is the same as that for $(2+x)^5$ except that the signs of the terms are alternately positive and negative instead of all positive. A similar remark applies to the expansion of every binomial of the form $(a-x)^n$ as compared to that of $(a+x)^n$.

EXERCISES

Expand each of the following powers.

1. $(x+y)^3$.
2. $(a+b)^4$.
3. $(x-y)^3$.
4. $(a-b)^4$.
5. $(2+r)^5$.
6. $(a+x)^7$.
7. $(g-3)^5$.
8. $(a^2+x)^5$.
9. $(a^2-x^2)^4$.
10. $(2\,a+1)^4$.
11. $(x-3\,y)^5$.
12. $(1+x^2)^6$.
13. $(1-x)^8$.
14. $(x-\frac{1}{2})^5$.
15. $(3\,a^2-1)^4$.
16. $(a+x)^{10}$.
17. $\left(\frac{1}{x}+\frac{1}{y}\right)^7$.
18. $\left(\frac{a}{x}-\frac{x}{a}\right)^5$.
19. $(\sqrt[3]{a^2}+\sqrt[4]{b^3})^3$.
20. $\left(\sqrt{2}+\frac{1}{x^2}\right)^3$

164. The General Term of $(a+x)^n$. The third term in the expansion of $(a+x)^n$, as given by the formula in § 163, is

$$\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2. \quad \textit{(third term)}$$

Observe that the exponent of x is 1 less than the number of the term; the exponent of a is n minus the exponent of x; the last factor of the denominator equals the exponent of x; in the numerator there are as many factors as in the denominator.

Precisely the same statements can be made as regards the fourth term, or

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3. \quad \textit{(fourth term)}$$

In the same way, it appears that the above statements can be made of *any* term, such as the rth, so that the formula for the rth term is

$$r\textit{th term} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\cdot a^{n-r+1}x^{r-1}.$$

EXAMPLE. Find the 7th term of $(2b-c)^{10}$.

SOLUTION. Here $a=2b$, $x=(-c)$, $n=10$, and $r=7$. Therefore (using the formula), the desired 7th term is

$$\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot(2b)^4(-c)^6=210(2b)^4(-c)^6=3360\,b^4c^6. \quad \textit{Ans.}$$

EXERCISES

Find each of the following indicated terms.

1. 5th term of $(a+x)^8$.
2. 6th term of $(x-y)^8$.
3. 7th term of $(2+x)^9$.
4. 10th term of $(m-n)^{14}$.
5. 6th term of $(a^2-b^2)^{10}$.
6. 20th term of $(1+x)^{24}$.
7. 6th term of $\left(x+\frac{1}{x}\right)^{11}$.
8. 9th term of $\left(\frac{a}{b}-b\right)^{16}$.
9. 5th term of $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^{12}$.
10. 4th term of $(2\sqrt{2}-\sqrt[3]{3})^6$.

165. Proof of the Binomial Theorem. The way in which the binomial formula was established in § 163 is, strictly speaking, open to objection because we there made sure of its correctness only for certain special values of n, such as $n=2$, $n=3$, $n=4$, and $n=5$. Though the formula holds

true, as we saw, in these cases, it does not follow necessarily that it is true in *every* case, that is for every positive integral value of n. We can now establish this fact, however, by the process of mathematical induction, when n is a positive integer.

Let m represent any special value of n for which the formula has been established (as, for example, 2, 3, 4, or 5). Then we have

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1\cdot 2}a^{m-2}x^2 + \cdots$$

$$(1) \qquad + \frac{m(m-1)\cdots(m-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+1}x^{r-1} + \cdots + x^m.$$

Let us now multiply both members of this equation by $a+x$. On the left we obtain $(a+x)^{m+1}$. On the right we shall have the sum of the two results obtained by multiplying the right side of (1) first by a and then by x, that is we shall have the sum of the two following expressions:

$$a^{m+1} + ma^m x + \frac{m(m-1)}{1\cdot 2}a^{m-1}x^2 + \cdots$$

$$+ \frac{m(m-1)\cdots(m-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+2}x^{r-1} + \cdots + ax^m,$$

and

$$a^m x + ma^{m-1}x^2 + \cdots + \frac{m(m-1)\cdots(m-r+3)}{1\cdot 2\cdot 3\cdots(r-2)}a^{m-r+2}x^{r-1}$$

$$+ \cdots + max^m + x^{m+1}.$$

Adding these, and making the natural simplifications in the resulting coefficients of $a^m x$, $a^{m-1}x^2$, etc., and equating the final result to its equal on the left (namely $(a+x)^{m+1}$, as noted above) gives

$$(a+x)^{m+1} = a^{m+1} + (m+1)a^m x + \frac{(m+1)m}{1\cdot 2}a^{m-1}x^2 + \cdots$$

$$(2) \qquad + \frac{(m+1)m\cdots(m-r+3)}{1\cdot 2\cdot 3\cdots(r-1)}a^{m-r+2}x^{r-1} + \cdots + x^{m+1}.$$

But (2) is precisely (1) except for the substitution of $m+1$ for m throughout. Hence, if the binomial formula holds for any special value of n, as m, it necessarily holds for the next larger value, namely $m+1$. But we have already observed that it holds when $n=5$. It must, therefore, hold when $n=5+1$, or 6. But if it holds when $n=6$, it must likewise hold when $n=6+1$, or 7. Thus we may proceed until we arrive at any chosen value of n whatever. That is, the formula must be true for *any* positive integral value of n.

***166. The Binomial Formula for Fractional and Negative Exponents.** In case the exponent n is *not* a positive integer but is fractional or negative, we may still write the expansion of $(a+x)^n$ by the formula of § 163, but it will now contain indefinitely many terms instead of coming to an end at some definite point, that is we meet with an *infinite series*. (Compare § 92.)

For example, the formula gives

$$(a+x)^{1/2}=a^{1/2}+\tfrac{1}{2}a^{1/2-1}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}a^{1/2-2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}a^{1/2-3}x^3+\cdots$$

$$=a^{1/2}+\tfrac{1}{2}\,a^{-1/2}x+\frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}\,a^{-3/2}x^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}\,a^{-5/2}x^3+\cdots$$

$$=a^{1/2}+\tfrac{1}{2}\,a^{-1/2}x-\tfrac{1}{8}\,a^{-3/2}x^2+\tfrac{1}{16}\,a^{-5/2}x^3+\cdots$$

Here we have written only the first four terms of the expansion, but we could obtain the 5th term in the same way and as many others in their order as might be desired.

***167. Application.** If in $(a+x)^n$ the value of x is small in comparison to that of a (more exactly, if the numerical value of x/a is less than 1) then the first few terms of the expansion furnish a close approximation to the value of $(a+x)^n$. This fact is often used to find approximate values for the roots of numbers in the manner illustrated below.

Example. Find the approximate value of $\sqrt{10}$.

Solution. Write $\sqrt{10}=\sqrt{9+1}=\sqrt{(3^2+1)}$ and expand this last form by the binomial formula. Thus (using the final result

in the worked example of § 166), we have

$$\sqrt{10}=(3^2+1)^{1/2}=(3^2)^{1/2}+\tfrac{1}{2}(3^2)^{-1/2}\cdot 1-\tfrac{1}{8}(3^2)^{-3/2}\cdot 1^2+\tfrac{1}{16}(3^2)^{-5/2}\cdot 1^3+\cdots$$

$$=3+\frac{1}{2\cdot 3}-\frac{1}{8\cdot 3^3}+\frac{1}{16\cdot 3^5}+\cdots,$$

$$=3+.166666-.004629+.000257=3.162288 \text{ (approximately)}.$$

Observe that the value of $\sqrt{10}$ as given in the tables is 3.16228, thus agreeing with that just found so far as the first five places of decimals are concerned.

Whenever extracting roots by this process we use the following general rule.

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the binomial theorem.

*EXERCISES

Write the first four terms in the expansion of each of the following expressions.

1. $(a+x)^{2/3}$.
2. $(a+x)^{-2}$.
3. $(1+x)^{1/3}$.
4. $(2-x)^{-1/4}$.
5. $(2a+b)^{3/4}$.
6. $(a^3-x^2)^{-3/4}$.
7. $\sqrt[5]{2+x}$.
8. $\sqrt[5]{a+x}$.
9. Find the 6th term in the expansion of $(a+x)^{1/2}$.

[Hint. Use the formula in § 164, with $n=\frac{1}{2}$ and $r=6$.]

Find the

10. 5th term of $(a+x)^{1/2}$.
11. 7th term of $(a+x)^{-2/3}$.
12. 8th term of $(1+x)^{1/3}$.
13. 9th term of $(a-x)^{-3}$.
14. 10th term of $\sqrt{(x+y)^3}$.
15. 6th term of $\sqrt[3]{2a+b}$.

Find the approximate values of the following to six decimal places and compare your results for the first three examples with those given in the tables.

16. $\sqrt{17}$.
17. $\sqrt{27}$.
18. $\sqrt[3]{9}$.
19. $\sqrt[4]{14}$.

[Hint. Write $14=16-2=2^4-2$.]

20. $\sqrt[5]{35}$.

CHAPTER XXIII

THE SOLUTION OF EQUATIONS BY DETERMINANTS

168. Definitions. The symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a ***determinant of the second order***, and is defined as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus

$$\begin{vmatrix} 8 & 3 \\ 2 & 4 \end{vmatrix} = 8 \cdot 4 - 2 \cdot 3 = 32 - 6 = 26.$$

$$\begin{vmatrix} 7 & 3 \\ -2 & 4 \end{vmatrix} = 7 \cdot 4 - (-2) \cdot 3 = 28 + 6 = 34.$$

$$\begin{aligned} 4\begin{vmatrix} -10 & 6 \\ -3 & 5 \end{vmatrix} &= 4[(-10) \cdot 5 - (-3) \cdot 6] = 4[-50 + 18] \\ &= 4(-32) = -128. \end{aligned}$$

The numbers a, b, c, and d are called the ***elements*** of the determinant.

The elements a and d (which lie along the diagonal through the upper left-hand corner of the determinant) form the ***principal diagonal***. The letters b and c (which lie along the diagonal through the upper right-hand corner) form the ***minor diagonal***.

From these definitions, we have the following rule.

To evaluate any determinant of the second order, subtract the product of the elements in the minor diagonal from the product of the elements in the principal diagonal.

EXERCISES

Evaluate each of the following determinants.

1. $\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$.

2. $\begin{vmatrix} 7 & 1 \\ 4 & -8 \end{vmatrix}$.

3. $\begin{vmatrix} 8 & 7 \\ -2 & 3 \end{vmatrix}$.

4. $\begin{vmatrix} -2 & 6 \\ 7 & -3 \end{vmatrix}$.

5. $\begin{vmatrix} 2a & 3b \\ 3a & 5b \end{vmatrix}$.

6. $5\begin{vmatrix} 4x & 6x \\ 3x^2 & 2x^2 \end{vmatrix}$.

7. $\frac{2}{3}\begin{vmatrix} 7a & 0 \\ 4b & 6b \end{vmatrix}$.

8. $\frac{3}{4}\begin{vmatrix} x+y & 3 \\ x-y & 4 \end{vmatrix}$.

169. Solution of Two Linear Equations. Let us consider a system of two linear equations between two unknown letters, x and y. Any such system is of the form

(1) $$a_1x+b_1y=c_1,$$

(2) $$a_2x+b_2y=c_2,$$

where a_1, b_1, c_1, etc., represent known numbers (coefficients).

This system may be solved for x and y by elimination, as in Chapter VII. Thus, multiplying (1) by b_2 and (2) by b_1 and then subtracting the resulting equations from each other, the letter y is eliminated and we reach the equation

$$(a_1b_2-a_2b_1)x=b_2c_1-b_1c_2.$$

Therefore

(3) $$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}.$$

Likewise, we may eliminate x by multiplying (1) by a_2 and

(2) by a_1 and subtracting the resulting equations from each other. This gives

$$(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1.$$

Therefore

$$y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}. \tag{4}$$

It is now clear, by § 168, that the numerators and denominators in (3) and (4) are all determinants of the second order; and by the definition of § 168, (3) and (4) may be written respectively in the forms

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{5}$$

These forms for the solution of (1) and (2) are easily remembered. In particular, observe that:

1. The determinant for the denominator is the same for both x and y.

2. The determinant for the numerator of the x-value is the same as that for the denominator except that the numbers c_1 and c_2 replace the a_1 and a_2 which occur in the *first* column of the denominator determinant.

3. The determinant for the numerator of the y-value is the same as that for the denominator except that the numbers c_1 and c_2 replace the b_1 and b_2 which occur in the *second* column of the denominator determinant.

The usefulness of the forms (5) lies in the fact that they express the solution of a system of two linear equations in condensed form, enabling us to write down the desired values of x and y *immediately*, without the usual process of elimination. This will now be illustrated.

EXAMPLE. Solve by determinants the system

$$(6) \qquad 2x+3y=18,$$

$$(7) \qquad x-7y=-8.$$

SOLUTION. Using the forms (5), we have at once

$$x=\frac{\begin{vmatrix}18 & 3\\ -8 & -7\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & -7\end{vmatrix}}=\frac{18\cdot(-7)-(-8)\cdot 3}{2(-7)-1\cdot 3}=\frac{-126+24}{-14-3}=\frac{-102}{-17}=6.$$

$$y=\frac{\begin{vmatrix}2 & 18\\ 1 & -8\end{vmatrix}}{\begin{vmatrix}2 & 3\\ 1 & -7\end{vmatrix}}=\frac{2\cdot(-8)-1\cdot 18}{2(-7)-1\cdot 3}=\frac{-16-18}{-14-3}=\frac{-34}{-17}=2.$$

The solution desired is therefore ($x=6$, $y=2$). *Ans.*

CHECK. Substituting 6 for x and 2 for y in (6) and (7) gives $12+6=18$ and $6-14=-8$, which are true results.

EXERCISES

Solve each of the following pairs of equations by determinants, checking your answers for each of the first three.

1. $\begin{cases} 6x+5y=3, \\ 8x+3y=-7. \end{cases}$

2. $\begin{cases} 3x-7y=-8, \\ 4x+3y=14. \end{cases}$

3. $\begin{cases} 3x+8y=0, \\ 2x-9y=-11. \end{cases}$

4. $\begin{cases} \frac{1}{2}x=\frac{1}{2}y+3, \\ \frac{2}{11}x+\frac{3}{4}y=16. \end{cases}$

5. $\begin{cases} .2a+.5b=30, \\ .4a-.8b=-16. \end{cases}$

6. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{8}=6\frac{1}{2}, \\ 2x-y=18. \end{cases}$

7. $\begin{cases} ax+by=r, \\ bx-ay=s. \end{cases}$

8. $\begin{cases} \dfrac{3ax+2by}{b}=a, \\ \dfrac{ax-by}{a}=b. \end{cases}$

***170. Determinants of the Third Order.** The symbol

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a ***determinant of the third order.***

Its value is defined as follows:

$$(2)\qquad a_1b_2c_3+b_1c_2a_3+c_1a_2b_3-a_3b_2c_1-b_3c_2a_1-c_3a_2b_1.$$

This expression, as we shall see presently, is important in the study of equations.

The expression (2) is called the *expanded form* of the determinant (1). It is important to observe that this expanded form may be written down at once as follows.

Write the determinant with the first two columns repeated at the right and first note the three diagonals which then run *down* from left to right (marked +). The product of the elements in the first of these diagonals is $a_1 b_2 c_3$, and this is seen to be the first term of the expanded form (2). Similarly, the product of the elements in the second of these diagonals is $b_1c_2a_3$, which forms the second term of (2); and likewise the third diagonal furnishes at once the third term of (2).

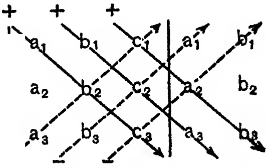


FIG. 78.

Next consider the three diagonals which run *up* from left to right (marked with dotted lines). The product of the elements in the first of these is $a_3 b_2 c_1$, and this is the fourth term of (2), provided it be taken negatively, that is preceded by the sign −. Similarly, the other two dotted diagonals of (3) furnish the last two terms of (2), provided they be taken negatively.

NOTE. Every determinant of the third order when expanded contains a total of *six* terms.

EXAMPLE. Expand and find the value of the determinant

$$\begin{vmatrix} 3 & 7 & 9 \\ 2 & 1 & 4 \\ 6 & 3 & 2 \end{vmatrix}.$$

SOLUTION. Repeating the first and second columns at the right, we have

$$\left|\begin{array}{ccc|cc} 3 & 7 & 9 & 3 & 7 \\ 2 & 1 & 4 & 2 & 1 \\ 6 & 3 & 2 & 6 & 3 \end{array}\right|.$$

The diagonals running down from left to right give the three products

$$3 \cdot 1 \cdot 2, \quad 7 \cdot 4 \cdot 6, \quad 9 \cdot 2 \cdot 3,$$

which form the first three terms of the expansion.

The diagonals running up from left to right give the products

$$6 \cdot 1 \cdot 9, \quad 3 \cdot 4 \cdot 3, \quad 2 \cdot 2 \cdot 7,$$

which, when taken negatively, form the three remaining terms of the determinant.

The complete expanded form of (3) is, therefore,

$$3 \cdot 1 \cdot 2 + 7 \cdot 4 \cdot 6 + 9 \cdot 2 \cdot 3 - 6 \cdot 1 \cdot 9 - 3 \cdot 4 \cdot 3 - 2 \cdot 2 \cdot 7,$$

which reduces to

$$6 + 168 + 54 - 54 - 36 - 28 = 110. \quad \textit{Ans.}$$

* EXERCISES

Expand and find the value of the following determinants.

1. $\begin{vmatrix} 1 & 3 & 7 \\ 2 & 4 & 6 \\ 3 & 5 & -4 \end{vmatrix}.$

2. $\begin{vmatrix} -7 & 2 & 2 \\ 3 & -4 & 6 \\ 8 & -5 & -3 \end{vmatrix}.$

3. $\begin{vmatrix} 8 & 2 & 3 \\ 6 & 0 & 5 \\ 3 & 0 & 7 \end{vmatrix}.$

4. $\begin{vmatrix} 2a & 3 & 6b \\ 3a & 2 & -5b \\ a & 0 & -2b \end{vmatrix}.$

5. $\begin{vmatrix} x & 7 & 1 \\ 2 & 3 & -4 \\ 4 & 2 & 1 \end{vmatrix}.$

6. $\begin{vmatrix} a & b & 2 \\ -4 & 5 & 3 \\ 2 & 1 & 0 \end{vmatrix}.$

7. $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix}.$

8. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & x-y & 0 \\ 0 & 0 & x+y \end{vmatrix}.$

***171. Solution of Three Linear Equations.** Let us consider a system of three linear equations between three unknown letters, such as x, y, and z. Any such system is of the form

$$(1)\qquad \begin{cases} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2, \\ a_3x+b_3y+c_3z=d_3, \end{cases}$$

where a_1, b_1, c_1, d_1, a_2, b_2, etc., represent known numbers (coefficients).

This system may be solved for x, y, and z by elimination, as in § 35, but the process is long. We shall here state merely the results, which are as follows (compare with (3) and (4) of § 169):

$$(2)\qquad \begin{cases} x=\dfrac{d_1b_2c_3+d_2b_3c_1+d_3b_1c_2-d_3b_2c_1-d_1b_3c_2-d_2b_1c_3}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_2b_1c_3}, \\ y=\dfrac{a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_1d_3c_2-a_3d_1c_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_3b_1c_2}, \\ z=\dfrac{a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_1b_3d_2-a_3b_1d_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_1b_3c_2-a_3b_1c_2}. \end{cases}$$

It is clear by § 170 that in these values for x, y, and z, each numerator and denominator is the expanded form of a determinant of the third order. In fact, it appears from the definition in § 170, that we may now express these values of x, y, and z in the following condensed (determinant) forms:

$$(3)\quad x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The importance of these expressions for x, y, and z lies in the fact that they give at once the solution of any system such as (1) in very compact and easily remembered forms. The following features should be especially noted:

1. The denominator determinant is the same in all three cases. (Compare statement 1 of § 169.)

2. The determinant for the numerator of the x-value is the same as that for the denominator determinant except that the numbers d_1, d_2, d_3 replace the a_1, a_2, a_3 which occur in the *first* column of the denominator determinant.

3. Similarly, the numerator of the y-value is formed from that of the denominator determinant by replacing the *second* column by the elements d_1, d_2, d_3; while the numerator of the z-value is formed from that of the denominator determinant by replacing the *third* column by the elements d_1, d_2, d_3. (Compare statements 2 and 3 of § 169.)

The readiness with which (3) may be used in practice to solve a system of three linear equations is illustrated by the following

EXAMPLE. Solve the system

$$\begin{cases} 2x-y+3z=35, \\ x+3y-15=-2z, \\ 3x+4y=1. \end{cases}$$

SOLUTION. Arranging the equations as in (1) of § 171, the given system is

$$\begin{aligned} 2x-y+3z&=35, \\ x+3y+2z&=15, \\ 3x+4y+0z&=1. \end{aligned}$$

Therefore, using (3) of § 171, we have at once

$$x=\frac{\begin{vmatrix} 35 & -1 & 3 \\ 15 & 3 & 2 \\ 1 & 4 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & 4 & 0 \end{vmatrix}}=\frac{0+180-2-9-280-0}{0+12-6-27-16-0}=\frac{-111}{-37}=3, \quad (\text{§ } 170)$$

$$y=\frac{\begin{vmatrix} 2 & 35 & 3 \\ 1 & 15 & 2 \\ 3 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & 4 & 0 \end{vmatrix}}=\frac{0+3+210-135-4-0}{-37}=\frac{74}{-37}=-2,$$

$$z=\frac{\begin{vmatrix} 2 & -1 & 35 \\ 1 & 3 & 15 \\ 3 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & 4 & 0 \end{vmatrix}}=\frac{6+140-45-315-120+1}{-37}=\frac{-333}{-37}=9.$$

The desired solution is, therefore, $(x=3,\ y=-2,\ z=9)$. *Ans.*

CHECK. With $x=3$, $y=-2$, $z=9$, it is readily seen that the three given equations are satisfied.

* EXERCISES

Solve each of the following systems by determinants.

1. $\begin{cases} x-2y+z=5, \\ 3x+6y-4z=3, \\ 8x-10y+3z=34. \end{cases}$

2. $\begin{cases} x-y+z=2, \\ x+y+z=6, \\ 2x+3y-4z=7. \end{cases}$

3. $\begin{cases} 2x-3y-4z=25, \\ x+y-z=-4, \\ 3x+y+2z=4. \end{cases}$

4. $\begin{cases} 4x-6y+2z=-9, \\ 3x+2y-z=2, \\ 2x-y+z=\frac{1}{2}. \end{cases}$

5. $\begin{cases} x+2y=0, \\ x+z=-3, \\ 4y+x-2z=4. \end{cases}$

6. $\begin{cases} \frac{x}{5}+\frac{y}{3}=7, \\ \frac{y}{5}+\frac{z}{3}=7, \\ \frac{x}{2}+\frac{y}{4}=8. \end{cases}$

7. $\begin{cases} x+y=3a, \\ x+z=5b, \\ y+z=2c. \end{cases}$

***172. Determinants of Higher Order.** Determinants of the fourth order exist and are studied in higher algebra, as are determinants of the fifth order, sixth order, etc. Moreover, determinants of the fourth order bear a similar relation to the solving of four linear equations between four unknown letters, as determinants of the third order bear to the solving of three linear equations between three unknown letters; and a similar remark may be made regarding determinants of the fifth order, sixth order, etc. In all cases, the solutions of such systems of equations can be expressed very simply by means of determinants.

T

APPENDIX

TABLE OF POWERS AND ROOTS

Explanation

1. Square Roots. The way to find square roots from the Table is best understood from an example. Thus, suppose we wish to find $\sqrt{1.48}$. To do this we first locate 1.48 in the column headed by the letter n. We find it near the bottom of this column (next to the last number). Now we go across on that level until we get into the column headed by $\sqrt{n}$. We find at that place the number 1.21655. This is our answer. That is, $\sqrt{1.48}=1.21655$ (approximately).

If we had wanted $\sqrt{14.8}$ instead of $\sqrt{1.48}$ the work would have been the same except that we would have gone over into the column headed $\sqrt{10\,n}$ (because $14.8=10\times 1.48$). The number thus located is seen to be 3.84708, which is, therefore, the desired value of $\sqrt{14.8}$.

Again, if we had wished to find $\sqrt{148}$ the work would take us back again to the column headed $\sqrt{n}$, but now instead of the answer being 1.21655 it would be 12.1655. In other words, the order of the digits in $\sqrt{148}$ is the same as for $\sqrt{1.48}$, but the decimal point in the answer is one place farther to the right.

Similarly, if we desired $\sqrt{1480}$ the work would be the same as before except that we must now use the column headed $\sqrt{10\,n}$ and move the decimal point there occurring one place farther to the right. This is seen to give 38.4708.

Thus we see how to get the square root of 1.48 or any power of 10 times that number.

In the same way, if we wish to find $\sqrt{.148}$, or $\sqrt{.0148}$, or $\sqrt{.00148}$, or the square root of any number obtained by dividing 1.48 by any power of 10 we can get the answers from the column headed $\sqrt{n}$ or $\sqrt{10\,n}$ by merely placing the decimal point properly. Thus, we find that $\sqrt{.148} = .384708$, $\sqrt{.0148} = .121655$, $\sqrt{.00148} = .0384708$, etc.

What we have seen in regard to the square root of 1.48 or of that number multiplied or divided by any power of 10 holds true in a similar way for *any* number that occurs in the column headed n, so that the tables thus give us the square roots of a great many numbers.

2. Cube Roots. Cube roots are located in the tables in much the same way as that just described for square roots, but we have here three columns to select from instead of two, namely the columns headed $\sqrt[3]{n}$, $\sqrt[3]{10\,n}$, $\sqrt[3]{100\,n}$.

Illustration.

$\sqrt[3]{1.48}$ occurs in the column headed $\sqrt[3]{n}$ and is seen to be 1.13960.
$\sqrt[3]{14.8}$ occurs in the column headed $\sqrt[3]{10\,n}$ and is seen to be 2.4552.
$\sqrt[3]{148}$ occurs in the column headed $\sqrt[3]{100\,n}$ and is seen to be 5.28957.

To get $\sqrt[3]{.148}$ we observe that $.148 = \sqrt[3]{\frac{1.48}{10}} = \sqrt[3]{\frac{148}{1000}} = \frac{1}{10}\sqrt[3]{148}$. Thus, we look up $\sqrt[3]{148}$ and divide it by 10. The result is instantly seen to be .528957. Similarly, to get $\sqrt[3]{.0148}$ we observe that $\sqrt[3]{.0148} = \sqrt[3]{\frac{1.48}{100}} = \sqrt[3]{\frac{14.8}{1000}} = \frac{1}{10}\sqrt[3]{14.8}$. Thus, we look up $\sqrt[3]{14.8}$ and divide it by 10, giving the result .24552.

To get $\sqrt[3]{.00148}$ we observe that $\sqrt[3]{.00148} = \sqrt{\frac{1.48}{1000}} = \frac{1}{10}\sqrt{1.48}$, so that we must divide $\sqrt{1.48}$ by 10. This gives .11396.

Similarly the cube root of any number occurring in the column headed n may be found, as well as the cube root of any number obtained by multiplying or dividing such a number by any power of 10.

3. Squares and Cubes. To find the square of 1.48 we naturally look at the proper level in the column headed n^2. Here we find 2.1904, which is the answer. If we wished the square of 14.8 the result would be the same except that the decimal point must be moved *two* places to the *right*, giving 219.04 as the answer. Similarly the value of $(148)^2$ is 21904.0 etc.

On the other hand, the value of $(.148)^2$ is found by moving the decimal place two places to the *left*, thus giving .021904. Similarly, $(.0148)^2 = .00021904$, etc.

To find $(1.48)^3$ we look at the proper level in the column headed n^3 where we find 3.24179. The value of $(14.8)^3$ is the same except that we must move the decimal point *three* places to the *right*, giving 3241.79. Similarly, in finding $(.148)^3$ we must move the decimal place three places to the *left*, giving .00324179.

Further illustrations of the way to use the tables will be found in § 43.

EXERCISES

Read off from the tables the values of each of the following expressions.

1. $\sqrt{41}$	**4.** $\sqrt[3]{670}$	**7.** $\sqrt{93.7}$	**10.** $\sqrt[3]{.00154}$
2. $\sqrt{8.9}$	**5.** $\sqrt{.89}$	**8.** $\sqrt[3]{93.7}$	**11.** $\sqrt{.000143}$
3. $\sqrt[3]{67}$	**6.** $\sqrt{.016}$	**9.** $\sqrt{.00154}$	**12.** $\sqrt[3]{.000143}$

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.00	1.0000	1.00000	3.16228	1.00000	1.00000	2.15443	4.64159
1.01	1.0201	1.00499	3.17805	1.03030	1.00332	2.16159	4.65701
1.02	1.0404	1.00995	3.19374	1.06121	1.00662	2.16870	4.67233
1.03	1.0609	1.01489	3.20936	1.09273	1.00990	2.17577	4.68755
1.04	1.0816	1.01980	3.22490	1.12486	1.01316	2.18279	4.70267
1.05	1.1025	1.02470	3.24037	1.15762	1.01640	2.18976	4.71769
1.06	1.1236	1.02956	3.25576	1.19102	1.01961	2.19669	4.73262
1.07	1.1449	1.03441	3.27109	1.22504	1.02281	2.20358	4.74746
1.08	1.1664	1.03923	3.28634	1.25971	1.02599	2.21042	4.76220
1.09	1.1881	1.04403	3.30151	1.29503	1.02914	2.21722	4.77686
1.10	1.2100	1.04881	3.31662	1.33100	1.03228	2.22398	4.79142
1.11	1.2321	1.05357	3.33167	1.36763	1.03540	2.23070	4.80590
1.12	1.2544	1.05830	3.34664	1.40493	1.03850	2.23738	4.82028
1.13	1.2769	1.06301	3.36155	1.44290	1.04158	2.24402	4.83459
1.14	1.2996	1.06771	3.37639	1.48154	1.04464	2.25062	4.84881
1.15	1.3225	1.07238	3.39116	1.52088	1.04769	2.25718	4.86294
1.16	1.3456	1.07703	3.40588	1.56090	1.05072	2.26370	4.87700
1.17	1.3689	1.08167	3.42053	1.60161	1.05373	2.27019	4.89097
1.18	1.3924	1.08628	3.43511	1.64303	1.05672	2.27664	4.90487
1.19	1.4161	1.09087	3.44964	1.68516	1.05970	2.28305	4.91868
1.20	1.4400	1.09545	3.46410	1.72800	1.06266	2.28943	4.93242
1.21	1.4641	1.10000	3.47851	1.77156	1.06560	2.29577	4.94609
1.22	1.4884	1.10454	3.49285	1.81585	1.06853	2.30208	4.95968
1.23	1.5129	1.10905	3.50714	1.86087	1.07144	2.30835	4.97319
1.24	1.5376	1.11355	3.52136	1.90662	1.07434	2.31459	4.98663
1.25	1.5625	1.11803	3.53553	1.95312	1.07722	2.32079	5.00000
1.26	1.5876	1.12250	3.54965	2.00038	1.08008	2.32697	5.01330
1.27	1.6129	1.12694	3.56371	2.04838	1.08293	2.33311	5.02653
1.28	1.6384	1.13137	3.57771	2.09715	1.08577	2.33921	5.03968
1.29	1.6641	1.13578	3.59166	2.14669	1.08859	2.34529	5.05277
1.30	1.6900	1.14018	3.60555	2.19700	1.09139	2.35133	5.06580
1.31	1.7161	1.14455	3.61939	2.24809	1.09418	2.35735	5.07875
1.32	1.7424	1.14891	3.63318	2.29997	1.09696	2.36333	5.09164
1.33	1.7689	1.15326	3.64692	2.35264	1.09972	2.36928	5.10447
1.34	1.7956	1.15758	3.66060	2.40610	1.10247	2.37521	5.11723
1.35	1.8225	1.16190	3.67423	2.46038	1.10521	2.38110	5.12993
1.36	1.8496	1.16619	3.68782	2.51546	1.10793	2.38697	5.14256
1.37	1.8769	1.17047	3.70135	2.57135	1.11064	2.39280	5.15514
1.38	1.9044	1.17473	3.71484	2.62807	1.11334	2.39861	5.16765
1.39	1.9321	1.17898	3.72827	2.68562	1.11602	2.40439	5.18010
1.40	1.9600	1.18322	3.74166	2.74400	1.11869	2.41014	5.19249
1.41	1.9881	1.18743	3.75500	2.80322	1.12135	2.41587	5.20483
1.42	2.0164	1.19164	3.76829	2.86329	1.12399	2.42156	5.21710
1.43	2.0449	1.19583	3.78153	2.92421	1.12662	2.42724	5.22932
1.44	2.0736	1.20000	3.79473	2.98598	1.12924	2.43288	5.24148
1.45	2.1025	1.20416	3.80789	3.04862	1.13185	2.43850	5.25359
1.46	2.1316	1.20830	3.82099	3.11214	1.13445	2.44409	5.26564
1.47	2.1609	1.21244	3.83406	3.17652	1.13703	2.44966	5.27763
1.48	2.1904	1.21655	3.84708	3.24179	1.13960	2.45520	5.28957
1.49	2.2201	1.22066	3.86005	3.30795	1.14216	2.46072	5.30146

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329
1.51	2.2801	1.22882	3.88587	3.44295	1.14725	2.47168	5.32507
1.52	2.3104	1.23288	3.89872	3.51181	1.14978	2.47712	5.33680
1.53	2.3409	1.23693	3.91152	3.58158	1.15230	2.48255	5.34848
1.54	2.3716	1.24097	3.92428	3.65226	1.15480	2.48794	5.36011
1.55	2.4025	1.24499	3.93700	3.72388	1.15729	2.49332	5.37169
1.56	2.4336	1.24900	3.94968	3.79642	1.15978	2.49867	5.38321
1.57	2.4649	1.25300	3.96232	3.86989	1.16225	2.50399	5.39469
1.58	2.4964	1.25698	3.97492	3.94431	1.16471	2.50930	5.40612
1.59	2.5281	1.26095	3.98748	4.01968	1.16717	2.51458	5.41750
1.60	2.5600	1.26491	4.00000	4.09600	1.16961	2.51984	5.42884
1.61	2.5921	1.26886	4.01248	4.17328	1.17204	2.52508	5.44012
1.62	2.6244	1.27279	4.02492	4.25153	1.17446	2.53030	5.45133
1.63	2.6569	1.27671	4.03733	4.33075	1.17687	2.53549	5.46256
1.64	2.6896	1.28062	4.04969	4.41094	1.17927	2.54067	5.47370
1.65	2.7225	1.28452	4.06202	4.49212	1.18167	2.54582	5.48481
1.66	2.7556	1.28841	4.07431	4.57430	1.18405	2.55095	5.49586
1.67	2.7889	1.29228	4.08656	4.65746	1.18642	2.55607	5.50688
1.68	2.8224	1.29615	4.09878	4.74163	1.18878	2.56116	5.51785
1.69	2.8561	1.30000	4.11096	4.82681	1.19114	2.56623	5.52877
1.70	2.8900	1.30384	4.12311	4.91300	1.19348	2.57128	5.53966
1.71	2.9241	1.30767	4.13521	5.00021	1.19582	2.57631	5.55050
1.72	2.9584	1.31149	4.14729	5.08845	1.19815	2.58133	5.56130
1.73	2.9929	1.31529	4.15933	5.17772	1.20046	2.58632	5.57205
1.74	3.0276	1.31909	4.17133	5.26802	1.20277	2.59129	5.58277
1.75	3.0625	1.32288	4.18330	5.35938	1.20507	2.59625	5.59344
1.76	3.0976	1.32665	4.19524	5.45178	1.20736	2.60118	5.60408
1.77	3.1329	1.33041	4.20714	5.54523	1.20964	2.60610	5.61467
1.78	3.1684	1.33417	4.21900	5.63975	1.21192	2.61100	5.62523
1.79	3.2041	1.33791	4.23084	5.73534	1.21418	2.61588	5.63574
1.80	3.2400	1.34164	4.24264	5.83200	1.21644	2.62074	5.64622
1.81	3.2761	1.34536	4.25441	5.92974	1.21869	2.62559	5.65665
1.82	3.3124	1.34907	4.26615	6.02857	1.22093	2.63041	5.66705
1.83	3.3489	1.35277	4.27785	6.12849	1.22316	2.63522	5.67741
1.84	3.3856	1.35647	4.28952	6.22950	1.22539	2.64001	5.68773
1.85	3.4225	1.36015	4.30116	6.33162	1.22760	2.64479	5.69802
1.86	3.4596	1.36382	4.31277	6.43486	1.22981	2.64954	5.70827
1.87	3.4969	1.36748	4.32435	6.53920	1.23201	2.65428	5.71848
1.88	3.5344	1.37113	4.33590	6.64467	1.23420	2.65901	5.72865
1.89	3.5721	1.37477	4.34741	6.75127	1.23639	2.66371	5.73879
1.90	3.6100	1.37840	4.35890	6.85900	1.23856	2.66840	5.74890
1.91	3.6481	1.38203	4.37035	6.96787	1.24073	2.67307	5.75897
1.92	3.6864	1.38564	4.38178	7.07789	1.24289	2.67773	5.76900
1.93	3.7249	1.38924	4.39318	7.18906	1.24505	2.68237	5.77900
1.94	3.7636	1.39284	4.40454	7.30138	1.24719	2.68700	5.78896
1.95	3.8025	1.39642	4.41588	7.41488	1.24933	2.69161	5.79889
1.96	3.8416	1.40000	4.42719	7.52954	1.25146	2.69620	5.80879
1.97	3.8809	1.40357	4.43847	7.64537	1.25359	2.70078	5.81865
1.98	3.9204	1.40712	4.44972	7.76239	1.25571	2.70534	5.82848
1.99	3.9601	1.41067	4.46094	7.88060	1.25782	2.70989	5.83827

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804
2.01	4.0401	1.41774	4.48330	8.12060	1.26202	2.71893	5.85777
2.02	4.0804	1.42127	4.49444	8.24241	1.26411	2.72344	5.86746
2.03	4.1209	1.42478	4.50555	8.36543	1.26619	2.72792	5.87713
2.04	4.1616	1.42829	4.51664	8.48966	1.26827	2.73239	5.88677
2.05	4.2025	1.43178	4.52769	8.61512	1.27033	2.73685	5.89637
2.06	4.2436	1.43527	4.53872	8.74182	1.27240	2.74129	5.90594
2.07	4.2849	1.43875	4.54973	8.86974	1.27445	2.74572	5.91548
2.08	4.3264	1.44222	4.56070	8.99891	1.27650	2.75014	5.92499
2.09	4.3681	1.44568	4.57165	9.12933	1.27854	2.75454	5.93447
2.10	4.4100	1.44914	4.58258	9.26100	1.28058	2.75892	5.94392
2.11	4.4521	1.45258	4.59347	9.39393	1.28261	2.76330	5.95334
2.12	4.4944	1.45602	4.60435	9.52813	1.28463	2.76766	5.96273
2.13	4.5369	1.45945	4.61519	9.66360	1.28665	2.77200	5.97209
2.14	4.5796	1.46287	4.62601	9.80034	1.28866	2.77633	5.98142
2.15	4.6225	1.46629	4.63681	9.93838	1.29066	2.78065	5.99073
2.16	4.6656	1.46969	4.64758	10.0777	1.29266	2.78495	6.00000
2.17	4.7089	1.47309	4.65833	10.2183	1.29465	2.78924	6.00925
2.18	4.7524	1.47648	4.66905	10.3602	1.29664	2.79352	6.01846
2.19	4.7961	1.47986	4.67974	10.5035	1.29862	2.79779	6.02765
2.20	4.8400	1.48324	4.69042	10.6480	1.30059	2.80204	6.03681
2.21	4.8841	1.48661	4.70106	10.7939	1.30256	2.80628	6.04594
2.22	4.9284	1.48997	4.71169	10.9410	1.30452	2.81050	6.05505
2.23	4.9729	1.49332	4.72229	11.0896	1.30648	2.81472	6.06413
2.24	5.0176	1.49666	4.73286	11.2394	1.30843	2.81892	6.07318
2.25	5.0625	1.50000	4.74342	11.3906	1.31037	2.82311	6.08220
2.26	5.1076	1.50333	4.75395	11.5432	1.31231	2.82728	6.09120
2.27	5.1529	1.50665	4.76445	11.6971	1.31424	2.83145	6.10017
2.28	5.1984	1.50997	4.77493	11.8524	1.31617	2.83560	6.10911
2.29	5.2441	1.51327	4.78539	12.0090	1.31809	2.83974	6.11803
2.30	5.2900	1.51658	4.79583	12.1670	1.32001	2.84387	6.12693
2.31	5.3361	1.51987	4.80625	12.3264	1.32192	2.84798	6.13579
2.32	5.3824	1.52315	4.81664	12.4872	1.32382	2.85209	6.14463
2.33	5.4289	1.52643	4.82701	12.6493	1.32572	2.85618	6.15345
2.34	5.4756	1.52971	4.83735	12.8129	1.32761	2.86026	6.16224
2.35	5.5225	1.53297	4.84768	12.9779	1.32950	2.86433	6.17101
2.36	5.5696	1.53623	4.85798	13.1443	1.33139	2.86838	6.17975
2.37	5.6169	1.53948	4.86826	13.3121	1.33326	2.87243	6.18846
2.38	5.6644	1.54272	4.87852	13.4813	1.33514	2.87646	6.19715
2.39	5.7121	1.54596	4.88876	13.6519	1.33700	2.88049	6.20582
2.40	5.7600	1.54919	4.89898	13.8240	1.33887	2.88450	6.21447
2.41	5.8081	1.55242	4.90918	13.9975	1.34072	2.88850	6.22308
2.42	5.8564	1.55563	4.91935	14.1725	1.34257	2.89249	6.23168
2.43	5.9049	1.55885	4.92950	14.3489	1.34442	2.89647	6.24025
2.44	5.9536	1.56205	4.93964	14.5268	1.34626	2.90044	6.24880
2.45	6.0025	1.56525	4.94975	14.7061	1.34810	2.90439	6.25732
2.46	6.0516	1.56844	4.95984	14.8869	1.34993	2.90834	6.26583
2.47	6.1009	1.57162	4.96991	15.0692	1.35176	2.91227	6.27431
2.48	6.1504	1.57480	4.97996	15.2530	1.35358	2.91620	6.28276
2.49	6.2001	1.57797	4.98999	15.4382	1.35540	2.92011	6.29119

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961
2.51	6.3001	1.58430	5.00999	15.8133	1.35902	2.92791	6.30799
2.52	6.3504	1.58745	5.01996	16.0030	1.36082	2.93179	6.31636
2.53	6.4009	1.59060	5.02991	16.1943	1.36262	2.93567	6.32470
2.54	6.4516	1.59374	5.03984	16.3871	1.36441	2.93953	6.33303
2.55	6.5025	1.59687	5.04975	16.5814	1.36620	2.94338	6.34133
2.56	6.5536	1.60000	5.05964	16.7772	1.36798	2.94723	6.34960
2.57	6.6049	1.60312	5.06952	16.9746	1.36976	2.95106	6.35786
2.58	6.6564	1.60624	5.07937	17.1735	1.37153	2.95488	6.36610
2.59	6.7081	1.60935	5.08920	17.3740	1.37330	2.95869	6.37431
2.60	6.7600	1.61245	5.09902	17.5760	1.37507	2.96250	6.38250
2.61	6.8121	1.61555	5.10882	17.7796	1.37683	2.96629	6.39068
2.62	6.8644	1.61864	5.11859	17.9847	1.37859	2.97007	6.39883
2.63	6.9169	1.62173	5.12835	18.1914	1.38034	2.97385	6.40696
2.64	6.9696	1.62481	5.13809	18.3997	1.38208	2.97761	6.41507
2.65	7.0225	1.62788	5.14782	18.6096	1.38383	2.98137	6.42316
2.66	7.0756	1.63095	5.15752	18.8211	1.38557	2.98511	6.43123
2.67	7.1289	1.63401	5.16720	19.0342	1.38730	2.98885	6.43928
2.68	7.1824	1.63707	5.17687	19.2488	1.38903	2.99257	6.44731
2.69	7.2361	1.64012	5.18652	19.4651	1.39076	2.99629	6.45531
2.70	7.2900	1.64317	5.19615	19.6830	1.39248	3.00000	6.46330
2.71	7.3441	1.64621	5.20577	19.9025	1.39419	3.00370	6.47127
2.72	7.3984	1.64924	5.21536	20.1236	1.39591	3.00739	6.47922
2.73	7.4529	1.65227	5.22494	20.3464	1.39761	3.01107	6.48715
2.74	7.5076	1.65529	5.23450	20.5708	1.39932	3.01474	6.49507
2.75	7.5625	1.65831	5.24404	20.7969	1.40102	3.01841	6.50296
2.76	7.6176	1.66132	5.25357	21.0246	1.40272	3.02206	6.51083
2.77	7.6729	1.66433	5.26308	21.2539	1.40441	3.02570	6.51868
2.78	7.7284	1.66733	5.27257	21.4850	1.40610	3.02934	6.52652
2.79	7.7841	1.67033	5.28205	21.7176	1.40778	3.03297	6.53434
2.80	7.8400	1.67332	5.29150	21.9520	1.40946	3.03659	6.54213
2.81	7.8961	1.67631	5.30094	22.1880	1.41114	3.04020	6.54991
2.82	7.9524	1.67929	5.31037	22.4258	1.41281	3.04380	6.55767
2.83	8.0089	1.68226	5.31977	22.6652	1.41448	3.04740	6.56541
2.84	8.0656	1.68523	5.32917	22.9063	1.41614	3.05098	6.57314
2.85	8.1225	1.68819	5.33854	23.1491	1.41780	3.05456	6.58084
2.86	8.1796	1.69115	5.34790	23.3937	1.41946	3.05813	6.58853
2.87	8.2369	1.69411	5.35724	23.6399	1.42111	3.06169	6.59620
2.88	8.2944	1.69706	5.36656	23.8879	1.42276	3.06524	6.60385
2.89	8.3521	1.70000	5.37587	24.1376	1.42440	3.06878	6.61149
2.90	8.4100	1.70294	5.38516	24.3890	1.42604	3.07232	6.61911
2.91	8.4681	1.70587	5.39444	24.6422	1.42768	3.07584	6.62671
2.92	8.5264	1.70880	5.40370	24.8971	1.42931	3.07936	6.63429
2.93	8.5849	1.71172	5.41295	25.1538	1.43094	3.08287	6.64185
2.94	8.6436	1.71464	5.42218	25.4122	1.43257	3.08638	6.64940
2.95	8.7025	1.71756	5.43139	25.6724	1.43419	3.08987	6.65693
2.96	8.7616	1.72047	5.44059	25.9343	1.43581	3.09336	6.66444
2.97	8.8209	1.72337	5.44977	26.1981	1.43743	3.09684	6.67194
2.98	8.8804	1.72627	5.45894	26.4636	1.43904	3.10031	6.67942
2.99	8.9401	1.72916	5.46809	26.7309	1.44065	3.10378	6.68688

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433
3.01	9.0601	1.73494	5.48635	27.2709	1.44385	3.11068	6.70176
3.02	9.1204	1.73781	5.49545	27.5436	1.44545	3.11412	6.70917
3.03	9.1809	1.74069	5.50454	27.8181	1.44704	3.11756	6.71657
3.04	9.2416	1.74356	5.51362	28.0945	1.44863	3.12098	6.72395
3.05	9.3025	1.74642	5.52268	28.3726	1.45022	3.12440	6.73132
3.06	9.3636	1.74929	5.53173	28.6526	1.45180	3.12781	6.73866
3.07	9.4249	1.75214	5.54076	28.9344	1.45338	3.13121	6.74600
3.08	9.4864	1.75499	5.54977	29.2181	1.45496	3.13461	6.75331
3.09	9.5481	1.75784	5.55878	29.5036	1.45653	3.13800	6.76061
3.10	9.6100	1.76068	5.56776	29.7910	1.45810	3.14138	6.76790
3.11	9.6721	1.76352	5.57674	30.0802	1.45967	3.14475	6.77517
3.12	9.7344	1.76635	5.58570	30.3713	1.46123	3.14812	6.78242
3.13	9.7969	1.76918	5.59464	30.6643	1.46279	3.15148	6.78966
3.14	9.8596	1.77200	5.60357	30.9591	1.46434	3.15483	6.79688
3.15	9.9225	1.77482	5.61249	31.2559	1.46590	3.15818	6.80409
3.16	9.9856	1.77764	5.62139	31.5545	1.46745	3.16152	6.81128
3.17	10.0489	1.78045	5.63028	31.8550	1.46899	3.16485	6.81846
3.18	10.1124	1.78326	5.63915	32.1574	1.47054	3.16817	6.82562
3.19	10.1761	1.78606	5.64801	32.4618	1.47208	3.17149	6.83277
3.20	10.2400	1.78885	5.65685	32.7680	1.47361	3.17480	6.83990
3.21	10.3041	1.79165	5.66569	33.0762	1.47515	3.17811	6.84702
3.22	10.3684	1.79444	5.67450	33.3862	1.47668	3.18140	6.85412
3.23	10.4329	1.79722	5.68331	33.6983	1.47820	3.18469	6.86121
3.24	10.4976	1.80000	5.69210	34.0122	1.47973	3.18798	6.86829
3.25	10.5625	1.80278	5.70088	34.3281	1.48125	3.19125	6.87534
3.26	10.6276	1.80555	5.70964	34.6460	1.48277	3.19452	6.88239
3.27	10.6929	1.80831	5.71839	34.9658	1.48428	3.19778	6.88942
3.28	10.7584	1.81108	5.72713	35.2876	1.48579	3.20104	6.89643
3.29	10.8241	1.81384	5.73585	35.6113	1.48730	3.20429	6.90344
3.30	10.8900	1.81659	5.74456	35.9370	1.48881	3.20753	6.91042
3.31	10.9561	1.81934	5.75326	36.2647	1.49031	3.21077	6.91740
3.32	11.0224	1.82209	5.76194	36.5944	1.49181	3.21400	6.92436
3.33	11.0889	1.82483	5.77062	36.9260	1.49330	3.21722	6.93130
3.34	11.1556	1.82757	5.77927	37.2597	1.49480	3.22044	6.93823
3.35	11.2225	1.83030	5.78792	37.5954	1.49629	3.22365	6.94515
3.36	11.2896	1.83303	5.79655	37.9331	1.49777	3.22686	6.95205
3.37	11.3569	1.83576	5.80517	38.2728	1.49926	3.23006	6.95894
3.38	11.4244	1.83848	5.81378	38.6145	1.50074	3.23325	6.96582
3.39	11.4921	1.84120	5.82237	38.9582	1.50222	3.23643	6.97268
3.40	11.5600	1.84391	5.83095	39.3040	1.50369	3.23961	6.97953
3.41	11.6281	1.84662	5.83952	39.6518	1.50517	3.24278	6.98637
3.42	11.6964	1.84932	5.84808	40.0017	1.50664	3.24595	6.99319
3.43	11.7649	1.85203	5.85662	40.3536	1.50810	3.24911	7.00000
3.44	11.8336	1.85472	5.86515	40.7076	1.50957	3.25227	7.00680
3.45	11.9025	1.85742	5.87367	41.0636	1.51103	3.25542	7.01358
3.46	11.9716	1.86011	5.88218	41.4217	1.51249	3.25856	7.02035
3.47	12.0409	1.86279	5.89067	41.7819	1.51394	3.26169	7.02711
3.48	12.1104	1.86548	5.89915	42.1442	1.51540	3.26482	7.03385
3.49	12.1801	1.86815	5.90762	42.5085	1.51685	3.26795	7.04058

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730
3.51	12.3201	1.87350	5.92453	43.2436	1.51974	3.27418	7.05400
3.52	12.3904	1.87617	5.93296	43.6142	1.52118	3.27729	7.06070
3.53	12.4609	1.87883	5.94138	43.9870	1.52262	3.28039	7.06738
3.54	12.5316	1.88149	5.94979	44.3619	1.52406	3.28348	7.07404
3.55	12.6025	1.88414	5.95819	44.7389	1.52549	3.28657	7.08070
3.56	12.6736	1.88680	5.96657	45.1180	1.52692	3.28965	7.08734
3.57	12.7449	1.88944	5.97495	45.4993	1.52835	3.29273	7.09397
3.58	12.8164	1.89209	5.98331	45.8827	1.52978	3.29580	7.10059
3.59	12.8881	1.89473	5.99166	46.2683	1.53120	3.29887	7.10719
3.60	12.9600	1.89737	6.00000	46.6560	1.53262	3.30193	7.11379
3.61	13.0321	1.90000	6.00833	47.0459	1.53404	3.30498	7.12037
3.62	13.1044	1.90263	6.01664	47.4379	1.53545	3.30803	7.12694
3.63	13.1769	1.90526	6.02495	47.8321	1.53686	3.31107	7.13349
3.64	13.2496	1.90788	6.03324	48.2285	1.53827	3.31411	7.14004
3.65	13.3225	1.91050	6.04152	48.6271	1.53968	3.31714	7.14657
3.66	13.3956	1.91311	6.04979	49.0279	1.54109	3.32017	7.15309
3.67	13.4689	1.91572	6.05805	49.4309	1.54249	3.32319	7.15960
3.68	13.5424	1.91833	6.06630	49.8360	1.54389	3.32621	7.16610
3.69	13.6161	1.92094	6.07454	50.2434	1.54529	3.32922	7.17258
3.70	13.6900	1.92354	6.08276	50.6530	1.54668	3.33222	7.17905
3.71	13.7641	1.92614	6.09098	51.0648	1.54807	3.33522	7.18552
3.72	13.8384	1.92873	6.09918	51.4788	1.54946	3.33822	7.19197
3.73	13.9129	1.93132	6.10737	51.8951	1.55085	3.34120	7.19840
3.74	13.9876	1.93391	6.11555	52.3136	1.55223	3.34419	7.20483
3.75	14.0625	1.93649	6.12372	52.7344	1.55362	3.34716	7.21125
3.76	14.1376	1.93907	6.13188	53.1574	1.55500	3.35014	7.21765
3.77	14.2129	1.94165	6.14003	53.5826	1.55637	3.35310	7.22405
3.78	14.2884	1.94422	6.14817	54.0102	1.55775	3.35607	7.23043
3.79	14.3641	1.94679	6.15630	54.4399	1.55912	3.35902	7.23680
3.80	14.4400	1.94936	6.16441	54.8720	1.56049	3.36198	7.24316
3.81	14.5161	1.95192	6.17252	55.3063	1.56186	3.36492	7.24950
3.82	14.5924	1.95448	6.18061	55.7430	1.56322	3.36786	7.25584
3.83	14.6689	1.95704	6.18870	56.1819	1.56459	3.37080	7.26217
3.84	14.7456	1.95959	6.19677	56.6231	1.56595	3.37373	7.26848
3.85	14.8225	1.96214	6.20484	57.0666	1.56731	3.37666	7.27479
3.86	14.8996	1.96469	6.21289	57.5125	1.56866	3.37958	7.28108
3.87	14.9769	1.96723	6.22093	57.9606	1.57001	3.38249	7.28736
3.88	15.0544	1.96977	6.22896	58.4111	1.57137	3.38540	7.29363
3.89	15.1321	1.97231	6.23699	58.8639	1.57271	3.38831	7.29989
3.90	15.2100	1.97484	6.24500	59.3190	1.57406	3.39121	7.30614
3.91	15.2881	1.97737	6.25300	59.7765	1.57541	3.39411	7.31238
3.92	15.3664	1.97990	6.26099	60.2363	1.57675	3.39700	7.31861
3.93	15.4449	1.98242	6.26897	60.6985	1.57809	3.39988	7.32483
3.94	15.5236	1.98494	6.27694	61.1630	1.57942	3.40277	7.33104
3.95	15.6025	1.98746	6.28490	61.6299	1.58076	3.40564	7.33723
3.96	15.6816	1.98997	6.29285	62.0991	1.58209	3.40851	7.34342
3.97	15.7609	1.99249	6.30079	62.5708	1.58342	3.41138	7.34960
3.98	15.8404	1.99499	6.30872	63.0448	1.58475	3.41424	7.35576
3.99	15.9201	1.99750	6.31664	63.5212	1.58608	3.41710	7.36192

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806
4.01	16.0801	2.00250	6.33246	64.4812	1.58872	3.42280	7.37420
4.02	16.1604	2.00499	6.34035	64.9648	1.59004	3.42564	7.38032
4.03	16.2409	2.00749	6.34823	65.4508	1.59136	3.42848	7.38644
4.04	16.3216	2.00998	6.35610	65.9393	1.59267	3.43131	7.39254
4.05	16.4025	2.01246	6.36396	66.4301	1.59399	3.43414	7.39864
4.06	16.4836	2.01494	6.37181	66.9234	1.59530	3.43697	7.40472
4.07	16.5649	2.01742	6.37966	67.4191	1.59661	3.43979	7.41080
4.08	16.6464	2.01990	6.38749	67.9173	1.59791	3.44260	7.41686
4.09	16.7281	2.02237	6.39531	68.4179	1.59922	3.44541	7.42291
4.10	16.8100	2.02485	6.40312	68.9210	1.60052	3.44822	7.42896
4.11	16.8921	2.02731	6.41093	69.4265	1.60182	3.45102	7.43499
4.12	16.9744	2.02978	6.41872	69.9345	1.60312	3.45382	7.44102
4.13	17.0569	2.03224	6.42651	70.4450	1.60441	3.45661	7.44703
4.14	17.1396	2.03470	6.43428	70.9579	1.60571	3.45939	7.45304
4.15	17.2225	2.03715	6.44205	71.4734	1.60700	3.46218	7.45904
4.16	17.3056	2.03961	6.44981	71.9913	1.60829	3.46496	7.46502
4.17	17.3889	2.04206	6.45755	72.5117	1.60958	3.46773	7.47100
4.18	17.4724	2.04450	6.46529	73.0346	1.61086	3.47050	7.47697
4.19	17.5561	2.04695	6.47302	73.5601	1.61215	3.47327	7.48292
4.20	17.6400	2.04939	6.48074	74.0880	1.61343	3.47603	7.48887
4.21	17.7241	2.05183	6.48845	74.6185	1.61471	3.47878	7.49481
4.22	17.8084	2.05426	6.49615	75.1514	1.61599	3.48154	7.50074
4.23	17.8929	2.05670	6.50384	75.6870	1.61726	3.48428	7.50666
4.24	17.9776	2.05913	6.51153	76.2250	1.61853	3.48703	7.51257
4.25	18.0625	2.06155	6.51920	76.7656	1.61981	3.48977	7.51847
4.26	18.1476	2.06398	6.52687	77.3088	1.62108	3.49250	7.52437
4.27	18.2329	2.06640	6.53452	77.8545	1.62234	3.49523	7.53025
4.28	18.3184	2.06882	6.54217	78.4028	1.62361	3.49796	7.53612
4.29	18.4041	2.07123	6.54981	78.9536	1.62487	3.50068	7.54199
4.30	18.4900	2.07364	6.55744	79.5070	1.62613	3.50340	7.54784
4.31	18.5761	2.07605	6.56506	80.0630	1.62739	3.50611	7.55369
4.32	18.6624	2.07846	6.57267	80.6216	1.62865	3.50882	7.55953
4.33	18.7489	2.08087	6.58027	81.1827	1.62991	3.51153	7.56535
4.34	18.8356	2.08327	6.58787	81.7465	1.63116	3.51423	7.57117
4.35	18.9225	2.08567	6.59545	82.3129	1.63241	3.51692	7.57698
4.36	19.0096	2.08806	6.60303	82.8819	1.63366	3.51962	7.58279
4.37	19.0969	2.09045	6.61060	83.4535	1.63491	3.52231	7.58858
4.38	19.1844	2.09284	6.61816	84.0277	1.63619	3.52499	7.59436
4.39	19.2721	2.09523	6.62571	84.6045	1.63740	3.52767	7.60014
4.40	19.3600	2.09762	6.63325	85.1840	1.63864	3.53035	7.60590
4.41	19.4481	2.10000	6.64078	85.7661	1.63988	3.53302	7.61166
4.42	19.5364	2.10238	6.64831	86.3509	1.64112	3.53569	7.61741
4.43	19.6249	2.10476	6.65582	86.9383	1.64236	3.53835	7.62315
4.44	19.7136	2.10713	6.66333	87.5284	1.64359	3.54101	7.62888
4.45	19.8025	2.10950	6.67083	88.1211	1.64483	3.54367	7.63461
4.46	19.8916	2.11187	6.67832	88.7165	1.64606	3.54632	7.64032
4.47	19.9809	2.11424	6.68581	89.3146	1.64729	3.54897	7.64603
4.48	20.0704	2.11660	6.69328	89.9154	1.64851	3.55162	7.65172
4.49	20.1601	2.11896	6.70075	90.5188	1.64974	3.55426	7.65741

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309
4.51	20.3401	2.12368	6.71565	91.7339	1.65219	3.55953	7.66877
4.52	20.4304	2.12603	6.72309	92.3454	1.65341	3.56215	7.67443
4.53	20.5209	2.12838	6.73053	92.9597	1.65462	3.56478	7.68009
4.54	20.6116	2.13073	6.73795	93.5767	1.65584	3.56740	7.68573
4.55	20.7025	2.13307	6.74537	94.1964	1.65706	3.57002	7.69137
4.56	20.7936	2.13542	6.75278	94.8188	1.65827	3.57263	7.69700
4.57	20.8849	2.13776	6.76018	95.4440	1.65948	3.57524	7.70262
4.58	20.9764	2.14009	6.76757	96.0719	1.66069	3.57785	7.70824
4.59	21.0681	2.14243	6.77495	96.7026	1.66190	3.58045	7.71384
4.60	21.1600	2.14476	6.78233	97.3360	1.66310	3.58305	7.71944
4.61	21.2521	2.14709	6.78970	97.9722	1.66431	3.58564	7.72503
4.62	21.3444	2.14942	6.79706	98.6111	1.66551	3.58823	7.73061
4.63	21.4369	2.15174	6.80441	99.2528	1.66671	3.59082	7.73619
4.64	21.5296	2.15407	6.81175	99.8973	1.66791	3.59340	7.74175
4.65	21.6225	2.15639	6.81909	100.545	1.66911	3.59598	7.74731
4.66	21.7156	2.15870	6.82642	101.195	1.67030	3.59856	7.75286
4.67	21.8089	2.16102	6.83374	101.848	1.67150	3.60113	7.75840
4.68	21.9024	2.16333	6.84105	102.503	1.67269	3.60370	7.76394
4.69	21.9961	2.16564	6.84836	103.162	1.67388	3.60626	7.76946
4.70	22.0900	2.16795	6.85565	103.823	1.67507	3.60883	7.77498
4.71	22.1841	2.17025	6.86294	104.487	1.67626	3.61138	7.78049
4.72	22.2784	2.17256	6.87023	105.154	1.67744	3.61394	7.78599
4.73	22.3729	2.17486	6.87750	105.824	1.67863	3.61649	7.79149
4.74	22.4676	2.17715	6.88477	106.496	1.67981	3.61903	7.79697
4.75	22.5625	2.17945	6.89202	107.172	1.68099	3.62158	7.80245
4.76	22.6576	2.18174	6.89928	107.850	1.68217	3.62412	7.80793
4.77	22.7529	2.18403	6.90652	108.531	1.68334	3.62665	7.81339
4.78	22.8484	2.18632	6.91375	109.215	1.68452	3.62919	7.81885
4.79	22.9441	2.18861	6.92098	109.902	1.68569	3.63172	7.82429
4.80	23.0400	2.19089	6.92820	110.592	1.68687	3.63424	7.82974
4.81	23.1361	2.19317	6.93542	111.285	1.68804	3.63676	7.83517
4.82	23.2324	2.19545	6.94262	111.980	1.68920	3.63928	7.84059
4.83	23.3289	2.19773	6.94982	112.679	1.69037	3.64180	7.84601
4.84	23.4256	2.20000	6.95701	113.380	1.69154	3.64431	7.85142
4.85	23.5225	2.20227	6.96419	114.084	1.69270	3.64682	7.85683
4.86	23.6196	2.20454	6.97137	114.791	1.69386	3.64932	7.86222
4.87	23.7169	2.20681	6.97854	115.501	1.69503	3.65182	7.86761
4.88	23.8144	2.20907	6.98570	116.214	1.69619	3.65432	7.87299
4.89	23.9121	2.21133	6.99285	116.930	1.69734	3.65681	7.87837
4.90	24.0100	2.21359	7.00000	117.649	1.69850	3.65931	7.88374
4.91	24.1081	2.21585	7.00714	118.371	1.69965	3.66179	7.88909
4.92	24.2064	2.21811	7.01427	119.095	1.70081	3.66428	7.89445
4.93	24.3049	2.22036	7.02140	119.823	1.70196	3.66676	7.89979
4.94	24.4036	2.22261	7.02851	120.554	1.70311	3.66924	7.90513
4.95	24.5025	2.22486	7.03562	121.287	1.70426	3.67171	7.91046
4.96	24.6016	2.22711	7.04273	122.024	1.70540	3.67418	7.91578
4.97	24.7009	2.22935	7.04982	122.763	1.70655	3.67665	7.92110
4.98	24.8004	2.23159	7.05691	123.506	1.70769	3.67911	7.92641
4.99	24.9001	2.23383	7.06399	124.251	1.70884	3.68157	7.93171

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701
5.01	25.1001	2.23830	7.07814	125.752	1.71112	3.68649	7.94229
5.02	25.2004	2.24054	7.08520	126.506	1.71225	3.68894	7.94757
5.03	25.3009	2.24277	7.09225	127.264	1.71339	3.69138	7.95285
5.04	25.4016	2.24499	7.09930	128.024	1.71452	3.69383	7.95811
5.05	25.5025	2.24722	7.10634	128.788	1.71566	3.69627	7.96337
5.06	25.6036	2.24944	7.11337	129.554	1.71679	3.69871	7.96863
5.07	25.7049	2.25167	7.12039	130.324	1.71792	3.70114	7.97387
5.08	25.8064	2.25389	7.12741	131.097	1.71905	3.70357	7.97911
5.09	25.9081	2.25610	7.13442	131.872	1.72017	3.70600	7.98434
5.10	26.0100	2.25832	7.14143	132.651	1.72130	3.70843	7.98957
5.11	26.1121	2.26053	7.14843	133.433	1.72242	3.71085	7.99479
5.12	26.2144	2.26274	7.15542	134.218	1.72355	3.71327	8.00000
5.13	26.3169	2.26495	7.16240	135.006	1.72467	3.71569	8.00520
5.14	26.4196	2.26716	7.16938	135.797	1.72579	3.71810	8.01040
5.15	26.5225	2.26936	7.17635	136.591	1.72691	3.72051	8.01559
5.16	26.6256	2.27156	7.18331	137.388	1.72802	3.72292	8.02078
5.17	26.7289	2.27376	7.19027	138.188	1.72914	3.72532	8.02596
5.18	26.8324	2.27596	7.19722	138.992	1.73025	3.72772	8.03113
5.19	26.9361	2.27816	7.20417	139.798	1.73137	3.73012	8.03629
5.20	27.0400	2.28035	7.21110	140.608	1.73248	3.73251	8.04145
5.21	27.1441	2.28254	7.21803	141.421	1.73359	3.73490	8.04660
5.22	27.2484	2.28473	7.22496	142.237	1.73470	3.73729	8.05175
5.23	27.3529	2.28692	7.23187	143.056	1.73580	3.73968	8.05689
5.24	27.4576	2.28910	7.23878	143.878	1.73691	3.74206	8.06202
5.25	27.5625	2.29129	7.24569	144.703	1.73801	3.74443	8.06714
5.26	27.6676	2.29347	7.25259	145.532	1.73912	3.74681	8.07226
5.27	27.7729	2.29565	7.25948	146.363	1.74022	3.74918	8.07737
5.28	27.8784	2.29783	7.26636	147.198	1.74132	3.75155	8.08248
5.29	27.9841	2.30000	7.27324	148.036	1.74242	3.75392	8.08758
5.30	28.0900	2.30217	7.28011	148.877	1.74351	3.75629	8.09267
5.31	28.1961	2.30434	7.28697	149.721	1.74461	3.75865	8.09776
5.32	28.3024	2.30651	7.29383	150.569	1.74570	3.76101	8.10284
5.33	28.4089	2.30868	7.30068	151.419	1.74680	3.76336	8.10791
5.34	28.5156	2.31084	7.30753	152.273	1.74789	3.76571	8.11298
5.35	28.6225	2.31301	7.31437	153.130	1.74898	3.76806	8.11804
5.36	28.7296	2.31517	7.32120	153.991	1.75007	3.77041	8.12310
5.37	28.8369	2.31733	7.32803	154.854	1.75116	3.77275	8.12814
5.38	28.9444	2.31948	7.33485	155.721	1.75224	3.77509	8.13319
5.39	29.0521	2.32164	7.34166	156.591	1.75333	3.77743	8.13822
5.40	29.1600	2.32379	7.34847	157.464	1.75441	3.77976	8.14325
5.41	29.2681	2.32594	7.35527	158.340	1.75549	3.78209	8.14828
5.42	29.3764	2.32809	7.36206	159.220	1.75657	3.78442	8.15329
5.43	29.4849	2.33024	7.36885	160.103	1.75765	3.78675	8.15831
5.44	29.5936	2.33238	7.37564	160.989	1.75873	3.78907	8.16331
5.45	29.7025	2.33452	7.38241	161.879	1.75981	3.79139	8.16831
5.46	29.8116	2.33666	7.38918	162.771	1.76088	3.79371	8.17330
5.47	29.9209	2.33880	7.39594	163.667	1.76196	3.79603	8.17829
5.48	30.0304	2.34094	7.40270	164.567	1.76303	3.79834	8.18327
5.49	30.1401	2.34307	7.40945	165.469	1.76410	3.80065	8.18824

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
5.51	30.3601	2.34734	7.42294	167.284	1.76624	3.80526	8.19818
5.52	30.4704	2.34947	7.42967	168.197	1.76731	3.80756	8.20313
5.53	30.5809	2.35160	7.43640	169.112	1.76838	3.80985	8.20808
5.54	30.6916	2.35372	7.44312	170.031	1.76944	3.81215	8.21303
5.55	30.8025	2.35584	7.44983	170.954	1.77051	3.81444	8.21797
5.56	30.9136	2.35797	7.45654	171.880	1.77157	3.81673	8.22290
5.57	31.0249	2.36008	7.46324	172.809	1.77263	3.81902	8.22783
5.58	31.1364	2.36220	7.46994	173.741	1.77369	3.82130	8.23275
5.59	31.2481	2.36432	7.47663	174.677	1.77475	3.82358	8.23766
5.60	31.3600	2.36643	7.48331	175.616	1.77581	3.82586	8.24257
5.61	31.4721	2.36854	7.48999	176.558	1.77686	3.82814	8.24747
5.62	31.5844	2.37065	7.49667	177.504	1.77792	3.83041	8.2523[illegible]
5.63	31.6969	2.37276	7.50333	178.454	1.77897	3.83268	8.25726
5.64	31.8096	2.37487	7.50999	179.406	1.78003	3.83495	8.26215
5.65	31.9225	2.37697	7.51665	180.362	1.78108	3.83722	8.26703
5.66	32.0356	2.37908	7.52330	181.321	1.78213	3.83948	8.27190
5.67	32.1489	2.38118	7.52994	182.284	1.78318	3.84174	8.27677
5.68	32.2624	2.38328	7.53658	183.250	1.78422	3.84399	8.28164
5.69	32.3761	2.38537	7.54321	184.220	1.78527	3.84625	8.28649
5.70	32.4900	2.38747	7.54983	185.193	1.78632	3.84850	8.29134
5.71	32.6041	2.38956	7.55645	186.169	1.78736	3.85075	8.29619
5.72	32.7184	2.39165	7.56307	187.149	1.78840	3.85300	8.30103
5.73	32.8329	2.39374	7.56968	188.133	1.78944	3.85524	8.30587
5.74	32.9476	2.39583	7.57628	189.119	1.79048	3.85748	8.31069
5.75	33.0625	2.39792	7.58288	190.109	1.79152	3.85972	8.31552
5.76	33.1776	2.40000	7.58947	191.103	1.79256	3.86196	8.32034
5.77	33.2929	2.40208	7.59605	192.100	1.79360	3.86419	8.32515
5.78	33.4084	2.40416	7.60263	193.101	1.79463	3.86642	8.32995
5.79	33.5241	2.40624	7.60920	194.105	1.79567	3.86865	8.33476
5.80	33.6400	2.40832	7.61577	195.112	1.79670	3.87088	8.33955
5.81	33.7561	2.41039	7.62234	196.123	1.79773	3.87310	8.34434
5.82	33.8724	2.41247	7.62889	197.137	1.79876	3.87532	8.34913
5.83	33.9889	2.41454	7.63544	198.155	1.79979	3.87754	8.35390
5.84	34.1056	2.41661	7.64199	199.177	1.80082	3.87975	8.35868
5.85	34.2225	2.41868	7.64853	200.202	1.80185	3.88197	8.36345
5.86	34.3396	2.42074	7.65506	201.230	1.80288	3.88418	8.36821
5.87	34.4569	2.42281	7.66159	202.262	1.80390	3.88639	8.37297
5.88	34.5744	2.42487	7.66812	203.297	1.80492	3.88859	8.37772
5.89	34.6921	2.42693	7.67463	204.336	1.80595	3.89080	8.38247
5.90	34.8100	2.42899	7.68115	205.379	1.80697	3.89300	8.38721
5.91	34.9281	2.43105	7.68765	206.425	1.80799	3.89519	8.39194
5.92	35.0464	2.43311	7.69415	207.475	1.80901	3.89739	8.39667
5.93	35.1649	2.43516	7.70065	208.528	1.81003	3.89958	8.40140
5.94	35.2836	2.43721	7.70714	209.585	1.81104	3.90177	8.40612
5.95	35.4025	2.43926	7.71362	210.645	1.81206	3.90396	8.41083
5.96	35.5216	2.44131	7.72010	211.709	1.81307	3.90615	8.41554
5.97	35.6409	2.44336	7.72658	212.776	1.81409	3.90833	8.42025
5.98	35.7604	2.44540	7.73305	213.847	1.81510	3.91051	8.42494
5.99	35.8801	2.44745	7.73951	214.922	1.81611	3.91269	8.42964

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433
6.01	36.1201	2.45153	7.75242	217.082	1.81813	3.91704	8.43901
6.02	36.2404	2.45357	7.75887	218.167	1.81914	3.91921	8.44369
6.03	36.3609	2.45561	7.76531	219.256	1.82014	3.92138	8.44836
6.04	36.4816	2.45764	7.77174	220.349	1.82115	3.92355	8.45303
6.05	36.6025	2.45967	7.77817	221.445	1.82215	3.92571	8.45769
6.06	36.7236	2.46171	7.78460	222.545	1.82316	3.92787	8.46235
6.07	36.8449	2.46374	7.79102	223.649	1.82416	3.93003	8.46700
6.08	36.9664	2.46577	7.79744	224.756	1.82516	3.93219	8.47165
6.09	37.0881	2.46779	7.80385	225.867	1.82616	3.93434	8.47629
6.10	37.2100	2.46982	7.81025	226.981	1.82716	3.93650	8.48093
6.11	37.3321	2.47184	7.81665	228.099	1.82816	3.93865	8.48556
6.12	37.4544	2.47386	7.82304	229.221	1.82915	3.94079	8.49018
6.13	37.5769	2.47588	7.82943	230.346	1.83015	3.94294	8.49481
6.14	37.6996	2.47790	7.83582	231.476	1.83115	3.94508	8.49942
6.15	37.8225	2.47992	7.84219	232.608	1.83214	3.94722	8.50403
6.16	37.9456	2.48193	7.84857	233.745	1.83313	3.94936	8.50864
6.17	38.0689	2.48395	7.85493	234.885	1.83412	3.95150	8.51324
6.18	38.1924	2.48596	7.86130	236.029	1.83511	3.95363	8.51784
6.19	38.3161	2.48797	7.86766	237.177	1.83610	3.95576	8.52243
6.20	38.4400	2.48998	7.87401	238.328	1.83709	3.95789	8.52702
6.21	38.5641	2.49199	7.88036	239.483	1.83808	3.96002	8.53160
6.22	38.6884	2.49399	7.88670	240.642	1.83906	3.96214	8.53618
6.23	38.8129	2.49600	7.89303	241.804	1.84005	3.96427	8.54075
6.24	38.9376	2.49800	7.89937	242.971	1.84103	3.96638	8.54532
6.25	39.0625	2.50000	7.90569	244.141	1.84202	3.96850	8.54988
6.26	39.1876	2.50200	7.91202	245.314	1.84300	3.97062	8.55444
6.27	39.3129	2.50400	7.91833	246.492	1.84398	3.97273	8.55899
6.28	39.4384	2.50599	7.92465	247.673	1.84496	3.97484	8.56354
6.29	39.5641	2.50799	7.93095	248.858	1.84594	3.97695	8.56808
6.30	39.6900	2.50998	7.93725	250.047	1.84691	3.97906	8.57262
6.31	39.8161	2.51197	7.94355	251.240	1.84789	3.98116	8.57715
6.32	39.9424	2.51396	7.94984	252.436	1.84887	3.98326	8.58168
6.33	40.0689	2.51595	7.95613	253.636	1.84984	3.98536	8.58620
6.34	40.1956	2.51794	7.96241	254.840	1.85082	3.98746	8.59072
6.35	40.3225	2.51992	7.96869	256.048	1.85179	3.98956	8.59524
6.36	40.4496	2.52190	7.97496	257.259	1.85276	3.99165	8.59975
6.37	40.5769	2.52389	7.98123	258.475	1.85373	3.99374	8.60425
6.38	40.7044	2.52587	7.98749	259.694	1.85470	3.99583	8.60875
6.39	40.8321	2.52784	7.99375	260.917	1.85567	3.99792	8.61325
6.40	40.9600	2.52982	8.00000	262.144	1.85664	4.00000	8.61774
6.41	41.0881	2.53180	8.00625	263.375	1.85760	4.00208	8.62222
6.42	41.2164	2.53377	8.01249	264.609	1.85857	4.00416	8.62671
6.43	41.3449	2.53574	8.01873	265.848	1.85953	4.00624	8.63118
6.44	41.4736	2.53772	8.02496	267.090	1.86050	4.00832	8.63566
6.45	41.6025	2.53969	8.03119	268.336	1.86146	4.01039	8.64012
6.46	41.7316	2.54165	8.03741	269.586	1.86242	4.01246	8.64459
6.47	41.8609	2.54362	8.04363	270.840	1.86338	4.01453	8.64904
6.48	41.9904	2.54558	8.04984	272.098	1.86434	4.01660	8.65350
6.49	42.1201	2.54755	8.05605	273.359	1.86530	4.01866	8.65795

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239
6.51	42.3801	2.55147	8.06846	275.894	1.86721	4.02279	8.66683
6.52	42.5104	2.55343	8.07465	277.168	1.86817	4.02485	8.67127
6.53	42.6409	2.55539	8.08084	278.445	1.86912	4.02690	8.67570
6.54	42.7716	2.55734	8.08703	279.726	1.87008	4.02896	8.68012
6.55	42.9025	2.55930	8.09321	281.011	1.87103	4.03101	8.68455
6.56	43.0336	2.56125	8.09938	282.300	1.87198	4.03306	8.68896
6.57	43.1649	2.56320	8.10555	283.593	1.87293	4.03511	8.69338
6.58	43.2964	2.56515	8.11172	284.890	1.87388	4.03715	8.69778
6.59	43.4281	2.56710	8.11788	286.191	1.87483	4.03920	8.70219
6.60	43.5600	2.56905	8.12404	287.496	1.87578	4.04124	8.70659
6.61	43.6921	2.57099	8.13019	288.805	1.87672	4.04328	8.71098
6.62	43.8244	2.57294	8.13634	290.118	1.87767	4.04532	8.71537
6.63	43.9569	2.57488	8.14248	291.434	1.87862	4.04735	8.71976
6.64	44.0896	2.57682	8.14862	292.755	1.87956	4.04939	8.72414
6.65	44.2225	2.57876	8.15475	294.080	1.88050	4.05142	8.72852
6.66	44.3556	2.58070	8.16088	295.408	1.88144	4.05345	8.73289
6.67	44.4889	2.58263	8.16701	296.741	1.88239	4.05548	8.73726
6.68	44.6224	2.58457	8.17313	298.078	1.88333	4.05750	8.74162
6.69	44.7561	2.58650	8.17924	299.418	1.88427	4.05953	8.74598
6.70	44.8900	2.58844	8.18535	300.763	1.88520	4.06155	8.75034
6.71	45.0241	2.59037	8.19146	302.112	1.88614	4.06357	8.75469
6.72	45.1584	2.59230	8.19756	303.464	1.88708	4.06559	8.75904
6.73	45.2929	2.59422	8.20366	304.821	1.88801	4.06760	8.76338
6.74	45.4276	2.59615	8.20975	306.182	1.88895	4.06961	8.76772
6.75	45.5625	2.59808	8.21584	307.547	1.88988	4.07163	8.77205
6.76	45.6976	2.60000	8.22192	308.916	1.89081	4.07364	8.77638
6.77	45.8329	2.60192	8.22800	310.289	1.89175	4.07564	8.78071
6.78	45.9684	2.60384	8.23408	311.666	1.89268	4.07765	8.78503
6.79	46.1041	2.60576	8.24015	313.047	1.89361	4.07965	8.78935
6.80	46.2400	2.60768	8.24621	314.432	1.89454	4.08166	8.79366
6.81	46.3761	2.60960	8.25227	315.821	1.89546	4.08365	8.79797
6.82	46.5124	2.61151	8.25833	317.215	1.89639	4.08565	8.80227
6.83	46.6489	2.61343	8.26438	318.612	1.89732	4.08765	8.80657
6.84	46.7856	2.61534	8.27043	320.014	1.89824	4.08964	8.81087
6.85	46.9225	2.61725	8.27647	321.419	1.89917	4.09163	8.81516
6.86	47.0596	2.61916	8.28251	322.829	1.90009	4.09362	8.81945
6.87	47.1969	2.62107	8.28855	324.243	1.90102	4.09561	8.82373
6.88	47.3344	2.62298	8.29458	325.661	1.90194	4.09760	8.82801
6.89	47.4721	2.62488	8.30060	327.083	1.90286	4.09958	8.83228
6.90	47.6100	2.62679	8.30662	328.509	1.90378	4.10157	8 83656
6.91	47.7481	2.62869	8.31264	329.939	1.90470	4.10355	8.84082
6.92	47.8864	2.63059	8.31865	331.374	1.90562	4.10552	8.84509
6.93	48.0249	2.63249	8.32466	332.813	1.90653	4.10750	8.84934
6.94	48.1636	2.63439	8.33067	334.255	1.90745	4.10948	8.85360
6.95	48.3025	2.63629	8.33667	335.702	1.90837	4.11145	8.85785
6.96	48.4416	2.63818	8.34266	337.154	1.90928	4.11342	8.86210
6.97	48.5809	2.64008	8.34865	338.609	1.91019	4.11539	8.86634
6.98	48.7204	2.64197	8.35464	340.068	1.91111	4.11736	8.87058
6.99	48.8601	2.64386	8.36062	341.532	1.91202	4.11932	8.87481

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904
7.01	49.1401	2.64764	8.37257	344.472	1.91384	4.12325	8.88327
7.02	49.2804	2.64953	8.37854	345.948	1.91475	4.12521	8.88749
7.03	49.4209	2.65141	8.38451	347.429	1.91566	4.12716	8.89171
7.04	49.5616	2.65330	8.39047	348.914	1.91657	4.12912	8.89592
7.05	49.7025	2.65518	8.39643	350.403	1.91747	4.13107	8.90013
7.06	49.8436	2.65707	8.40238	351.896	1.91838	4.13303	8.90434
7.07	49.9849	2.65895	8.40833	353.393	1.91929	4.13498	8.90854
7.08	50.1264	2.66083	8.41427	354.895	1.92019	4.13693	8.91274
7.09	50.2681	2.66271	8.42021	356.401	1.92109	4.13887	8.91693
7.10	50.4100	2.66458	8.42615	357.911	1.92200	4.14082	8.92112
7.11	50.5521	2.66646	8.43208	359.425	1.92290	4.14276	8.92531
7.12	50.6944	2.66833	8.43801	360.944	1.92380	4.14470	8.92949
7.13	50.8369	2.67021	8.44393	362.467	1.92470	4.14664	8.93367
7.14	50.9796	2.67208	8.44985	363.994	1.92560	4.14858	8.93784
7.15	51.1225	2.67395	8.45577	365.526	1.92650	4.15052	8.94201
7.16	51.2656	2.67582	8.46168	367.062	1.92740	4.15245	8.94618
7.17	51.4089	2.67769	8.46759	368.602	1.92829	4.15438	8.95034
7.18	51.5524	2.67955	8.47349	370.146	1.92919	4.15631	8.95450
7.19	51.6961	2.68142	8.47939	371.695	1.93008	4.15824	8.95866
7.20	51.8400	2.68328	8.48528	373.248	1.93098	4.16017	8.96281
7.21	51.9841	2.68514	8.49117	374.805	1.93187	4.16209	8.96696
7.22	52.1284	2.68701	8.49706	376.367	1.93277	4.16402	8.97110
7.23	52.2729	2.68887	8.50294	377.933	1.93366	4.16594	8.97524
7.24	52.4176	2.69072	8.50882	379.503	1.93455	4.16786	8.97938
7.25	52.5625	2.69258	8.51469	381.078	1.93544	4.16978	8.98351
7.26	52.7076	2.69444	8.52056	382.657	1.93633	4.17169	8.98764
7.27	52.8529	2.69629	8.52643	384.241	1.93722	4.17361	8.99176
7.28	52.9984	2.69815	8.53229	385.828	1.93810	4.17552	8.99588
7.29	53.1441	2.70000	8.53815	387.420	1.93899	4.17743	9 00000
7.30	53.2900	2.70185	8.54400	389 017	1.93988	4.17934	9.00411
7.31	53.4361	2.70370	8.54985	390.618	1.94076	4.18125	9.00822
7.32	53.5824	2.70555	8.55570	392.223	1.94165	4.18315	9.01233
7.33	53.7289	2.70740	8.56154	393.833	1.94253	4.18506	9.01643
7.34	53.8756	2.70924	8.56738	395.447	1.94341	4.18696	9.02053
7.35	54.0225	2.71109	8.57321	397.065	1.94430	4.18886	9.02462
7.36	54.1696	2.71293	8.57904	398.688	1.94518	4.19076	9.02871
7.37	54.3169	2.71477	8.58487	400.316	1.94606	4.19266	9.03280
7.38	54.4644	2.71662	8.59069	401.947	1.94694	4.19455	9.03689
7.39	54.6121	2.71846	8.59651	403.583	1.94782	4.19644	9.04097
7.40	54.7600	2.72029	8.60233	405.224	1.94870	4.19834	9.04504
7.41	54.9081	2.72213	8.60814	406.869	1.94957	4.20023	9.04911
7.42	55.0564	2.72397	8.61394	408.518	1.95045	4.20212	9.05318
7.43	55.2049	2.72580	8.61974	410.172	1.95132	4.20400	9.05725
7.44	55.3536	2.72764	8.62554	411.831	1.95220	4.20589	9.06131
7.45	55.5025	2.72947	8.63134	413.494	1.95307	4.20777	9.06537
7.46	55.6516	2.73130	8.63713	415.161	1.95395	4.20965	9.06942
7.47	55.8009	2.73313	8.64292	416.833	1.95482	4.21153	9.07347
7.48	55.9504	2.73496	8.64870	418.509	1.95569	4.21341	9.07752
7.49	56.1001	2.73679	8.65448	420.190	1.95656	4.21529	9.08156

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560
7.51	56.4001	2.74044	8.66603	423.565	1.95830	4.21904	9.08964
7.52	56.5504	2.74226	8.67179	425.259	1.95917	4.22091	9.09367
7.53	56.7009	2.74408	8.67756	426.958	1.96004	4.22278	9.09770
7.54	56.8516	2.74591	8.68332	428.661	1.96091	4.22465	9.10173
7.55	57.0025	2.74773	8.68907	430.369	1.96177	4.22651	9.10575
7.56	57.1536	2.74955	8.69483	432.081	1.96264	4.22838	9.10977
7.57	57.3049	2.75136	8.70057	433.798	1.96350	4.23024	9.11378
7.58	57.4564	2.75318	8.70632	435.520	1.96437	4.23210	9.11779
7.59	57.6081	2.75500	8.71206	437.245	1.96523	4.23396	9.12180
7.60	57.7600	2.75681	8.71780	438.976	1.96610	4.23582	9.12581
7.61	57.9121	2.75862	8.72353	440.711	1.96696	4.23768	9.12981
7.62	58.0644	2.76043	8.72926	442.451	1.96782	4.23954	9.13380
7.63	58.2169	2.76225	8.73499	444.195	1.96868	4.24139	9.13780
7.64	58.3696	2.76405	8.74071	445.944	1.96954	4.24324	9.14179
7.65	58.5225	2.76586	8.74643	447.697	1.97040	4.24509	9.14577
7.66	58.6756	2.76767	8.75214	449.455	1.97126	4.24694	9.14976
7.67	58.8289	2.76948	8.75785	451.218	1.97211	4.24879	9.15374
7.68	58.9824	2.77128	8.76356	452.985	1.97297	4.25063	9.15771
7.69	59.1361	2.77308	8.76926	454.757	1.97383	4.25248	9.16169
7.70	59.2900	2.77489	8.77496	456.533	1.97468	4.25432	9.16566
7.71	59.4441	2.77669	8.78066	458.314	1.97554	4.25616	9.16962
7.72	59.5984	2.77849	8.78635	460.100	1.97639	4.25800	9.17359
7.73	59.7529	2.78029	8.79204	461.890	1.97724	4.25984	9.17754
7.74	59.9076	2.78209	8.79773	463.685	1.97809	4.26167	9.18150
7.75	60.0625	2.78388	8.80341	465.484	1.97895	4.26351	9.18545
7.76	60.2176	2.78568	8.80909	467.289	1.97980	4.26534	9.18940
7.77	60.3729	2.78747	8.81476	469.097	1.98065	4.26717	9.19335
7.78	60.5284	2.78927	8.82043	470.911	1.98150	4.26900	9.19729
7.79	60.6841	2.79106	8.82610	472.729	1.98234	4.27083	9.20123
7.80	60.8400	2.79285	8.83176	474.552	1.98319	4.27266	9.20516
7.81	60.9961	2.79464	8.83742	476.380	1.98404	4.27448	9.20910
7.82	61.1524	2.79643	8.84308	478.212	1.98489	4.27631	9.21302
7.83	61.3089	2.79821	8.84873	480.049	1.98573	4.27813	9.21695
7.84	61.4656	2.80000	8.85438	481.890	1.98658	4.27995	9.22087
7.85	61.6225	2.80179	8.86002	483.737	1.98742	4.28177	9.22479
7.86	61.7796	2.80357	8.86566	485.588	1.98826	4.28359	9.22871
7.87	61.9369	2.80535	8.87130	487.443	1.98911	4.28540	9.23262
7.88	62.0944	2.80713	8.87694	489.304	1.98995	4.28722	9.23653
7.89	62.2521	2.80891	8.88257	491.169	1.99079	4.28903	9.24043
7.90	62.4100	2.81069	8.88819	493.039	1.99163	4.29084	9.24434
7.91	62.5681	2.81247	8.89382	494.914	1.99247	4.29265	9.24823
7.92	62.7264	2.81425	8.89944	496.793	1.99331	4.29446	9.25213
7.93	62.8849	2.81603	8.90505	498.677	1.99415	4.29627	9.25602
7.94	63.0436	2.81780	8.91067	500.566	1.99499	4.29807	9.25991
7.95	63.2025	2.81957	8.91628	502.460	1.99582	4.29987	9.26380
7.96	63.3616	2.82135	8.92188	504.358	1.99666	4.30168	9.26768
7.97	63.5209	2.82312	8.92749	506.262	1.99750	4.30348	9.27156
7.98	63.6804	2.82489	8.93308	508.170	1.99833	4.30528	9.27544
7.99	63.8401	2.82666	8.93868	510.082	1.99917	4.30707	9.27931

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318
8.01	64.1601	2.83019	8.94986	513.922	2.00083	4.31066	9.28704
8.02	64.3204	2.83196	8.95545	515.850	2.00167	4.31246	9.29091
8.03	64.4809	2.83373	8.96103	517.782	2.00250	4.31425	9.29477
8.04	64.6416	2.83549	8.96660	519.718	2.00333	4.31604	9.29862
8.05	64.8025	2.83725	8.97218	521.660	2.00416	4.31783	9.30248
8.06	64.9636	2.83901	8.97775	523.607	2.00499	4.31961	9.30633
8.07	65.1249	2.84077	8.98332	525.558	2.00582	4.32140	9.31018
8.08	65.2864	2.84253	8.98888	527.514	2.00664	4.32318	9.31402
8.09	65.4481	2.84429	8.99444	529.475	2.00747	4.32497	9.31786
8.10	65.6100	2.84605	9.00000	531.441	2.00830	4.32675	9.32170
8.11	65.7721	2.84781	9.00555	533.412	2.00912	4.32853	9.32553
8.12	65.9344	2.84956	9.01110	535.387	2.00995	4.33031	9.32936
8.13	66.0969	2.85132	9.01665	537.368	2.01078	4.33208	9.33319
8.14	66.2596	2.85307	9.02219	539.353	2.01160	4.33386	9.33702
8.15	66.4225	2.85482	9.02774	541.343	2.01242	4.33563	9.34084
8.16	66.5856	2.85657	9.03327	543.338	2.01325	4.33741	9.34466
8.17	66.7489	2.85832	9.03881	545.339	2.01407	4.33918	9.34847
8.18	66.9124	2.86007	9.04434	547.343	2.01489	4.34095	9.35229
8.19	67.0761	2.86182	9.04986	549.353	2.01571	4.34271	9.35610
8.20	67.2400	2.86356	9.05539	551.368	2.01653	4.34448	9.35990
8.21	67.4041	2.86531	9.06091	553.388	2.01735	4.34625	9.36370
8.22	67.5684	2.86705	9.06642	555.412	2.01817	4.34801	9.36751
8.23	67.7329	2.86880	9.07193	557.442	2.01899	4.34977	9.37130
8.24	67.8976	2.87054	9.07744	559.476	2.01980	4.35153	9.37510
8.25	68.0625	2.87228	9.08295	561.516	2.02062	4.35329	9.37889
8.26	68.2276	2.87402	9.08845	563.560	2.02144	4.35505	9.38268
8.27	68.3929	2.87576	9.09395	565.609	2.02225	4.35681	9.38646
8.28	68.5584	2.87750	9.09945	567.664	2.02307	4.35856	9.39024
8.29	68.7241	2.87924	9.10494	569.723	2.02388	4.36032	9.39402
8.30	68.8900	2.88097	9.11043	571.787	2.02469	4.36207	9.39780
8.31	69.0561	2.88271	9.11592	573.856	2.02551	4.36382	9.40157
8.32	69.2224	2.88444	9.12140	575.930	2.02632	4.36557	9.40534
8.33	69.3889	2.88617	9.12688	578.010	2.02713	4.36732	9.40911
8.34	69.5556	2.88791	9.13236	580.094	2.02794	4.36907	9.41287
8.35	69.7225	2.88964	9.13783	582.183	2.02875	4.37081	9.41663
8.36	69.8896	2.89137	9.14330	584.277	2.02956	4.37256	9.42039
8.37	70.0569	2.89310	9.14877	586.376	2.03037	4.37430	9.42414
8.38	70.2244	2.89482	9.15423	588.480	2.03118	4.37604	9.42789
8.39	70.3921	2.89655	9.15969	590.590	2.03199	4.37778	9.43164
8.40	70.5600	2.89828	9.16515	592.704	2.03279	4.37952	9.43539
8.41	70.7281	2.90000	9.17061	594.823	2.03360	4.38126	9.43913
8.42	70.8964	2.90172	9.17606	596.948	2.03440	4.38299	9.44287
8.43	71.0649	2.90345	9.18150	599.077	2.03521	4.38473	9.44661
8.44	71.2336	2.90517	9.18695	601.212	2.03601	4.38646	9.45034
8.45	71.4025	2.90689	9.19239	603.351	2.03682	4.38819	9.45407
8.46	71.5716	2.90861	9.19783	605.496	2.03762	4.38992	9.45780
8.47	71.7409	2.91033	9.20326	607.645	2.03842	4.39165	9.46152
8.48	71.9104	2.91204	9.20869	609.800	2.03923	4.39338	9.46525
8.49	72.0801	2.91376	9.21412	611.960	2.04003	4.39510	9.46897

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268
8.51	72.4201	2.91719	9.22497	616.295	2.04163	4.39855	9.47640
8.52	72.5904	2.91890	9.23038	618.470	2.04243	4.40028	9.48011
8.53	72.7609	2.92062	9.23580	620.650	2.04323	4.40200	9.48381
8.54	72.9316	2.92233	9.24121	622.836	2.04402	4.40372	9.48752
8.55	73.1025	2.92404	9.24662	625.026	2.04482	4.40543	9.49122
8.56	73.2736	2.92575	9.25203	627.222	2.04562	4.40715	9.49492
8.57	73.4449	2.92746	9.25743	629.423	2.04641	4.40887	9.49861
8.58	73.6164	2.92916	9.26283	631.629	2.04721	4.41058	9.50231
8.59	73.7881	2.93087	9.26823	633.840	2.04801	4.41229	9.50600
8.60	73.9600	2.93258	9.27362	636.056	2.04880	4.41400	9.50969
8.61	74.1321	2.93428	9.27901	638.277	2.04959	4.41571	9.51337
8.62	74.3044	2.93598	9.28440	640.504	2.05039	4.41742	9.51705
8.63	74.4769	2.93769	9.28978	642.736	2.05118	4.41913	9.52073
8.64	74.6496	2.93939	9.29516	644.973	2.05197	4.42084	9.52441
8.65	74.8225	2.94109	9.30054	647.215	2.05276	4.42254	9.52808
8.66	74.9956	2.94279	9.30591	649.462	2.05355	4.42425	9.53175
8.67	75.1689	2.94449	9.31128	651.714	2.05434	4.42595	9.53542
8.68	75.3424	2.94618	9.31665	653.972	2.05513	4.42765	9.53908
8.69	75.5161	2.94788	9.32202	656.235	2.05592	4.42935	9.54274
8.70	75.6900	2.94958	9.32738	658.503	2.05671	4.43105	9.54640
8.71	75.8641	2.95127	9.33274	660.776	2.05750	4.43274	9.55006
8.72	76.0384	2.95296	9.33809	663.055	2.05828	4.43444	9.55371
8.73	76.2129	2.95466	9.34345	665.339	2.05907	4.43613	9.55736
8.74	76.3876	2.95635	9.34880	667.628	2.05986	4.43783	9.56101
8.75	76.5625	2.95804	9.35414	669.922	2.06064	4.43952	9.56466
8.76	76.7376	2.95973	9.35949	672.221	2.06143	4.44121	9.56830
8.77	76.9129	2.96142	9.36483	674.526	2.06221	4.44290	9.57194
8.78	77.0884	2.96311	9.37017	676.836	2.06299	4.44459	9.57557
8.79	77.2641	2.96479	9.37550	679.151	2.06378	4.44627	9.57921
8.80	77.4400	2.96648	9.38083	681.472	2.06456	4.44796	9.58284
8.81	77.6161	2.96816	9.38616	683.798	2.06534	4.44964	9.58647
8.82	77.7924	2.96985	9.39149	686.129	2.06612	4.45133	9.59009
8.83	77.9689	2.97153	9.39681	688.465	2.06690	4.45301	9.59372
8.84	78.1456	2.97321	9.40213	690.807	2.06768	4.45469	9.59734
8.85	78.3225	2.97489	9.40744	693.154	2.06846	4.45637	9.60095
8.86	78.4996	2.97658	9.41276	695.506	2.06924	4.45805	9.60457
8.87	78.6769	2.97825	9.41807	697.864	2.07002	4.45972	9.60818
8.88	78.8544	2.97993	9.42338	700.227	2.07080	4.46140	9.61179
8.89	79.0321	2.98161	9.42868	702.595	2.07157	4.46307	9.61540
8.90	79.2100	2.98329	9.43398	704.969	2.07235	4.46475	9.61900
8.91	79.3881	2.98496	9.43928	707.348	2.07313	4.46642	9.62260
8.92	79.5664	2.98664	9.44458	709.732	2.07390	4.46809	9.62620
8.93	79.7449	2.98831	9.44987	712.122	2.07468	4.46976	9.62980
8.94	79.9236	2.98998	9.45516	714.517	2.07545	4.47142	9.63339
8.95	80.1025	2.99166	9.46044	716.917	2.07622	4.47309	9.63698
8.96	80.2816	2.99333	9.46573	719.323	2.07700	4.47476	9.64057
8.97	80.4609	2.99500	9.47101	721.734	2.07777	4.47642	9.64415
8.98	80.6404	2.99666	9.47629	724.151	2.07854	4.47808	9.64774
8.99	80.8201	2.99833	9.48156	726.573	2.07931	4.47974	9.65132

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489
9.01	81.1801	3.00167	9.49210	731.433	2.08085	4.48306	9.65847
9.02	81.3604	3.00333	9.49737	733.871	2.08162	4.48472	9.66204
9.03	81.5409	3.00500	9.50263	736.314	2.08239	4.48638	9.66561
9.04	81.7216	3.00666	9.50789	738.763	2.08316	4.48803	9.66918
9.05	81.9025	3.00832	9.51315	741.218	2.08393	4.48969	9.67274
9.06	82.0836	3.00998	9.51840	743.677	2.08470	4.49134	9.67630
9.07	82.2649	3.01164	9.52365	746.143	2.08546	4.49299	9.67986
9.08	82.4464	3.01330	9.52890	748.613	2.08623	4.49464	9.68342
9.09	82.6281	3.01496	9.53415	751.089	2.08699	4.49629	9.68697
9.10	82.8100	3.01662	9.53939	753.571	2.08776	4.49794	9.69052
9.11	82.9921	3.01828	9.54463	756.058	2.08852	4.49959	9.69407
9.12	83.1744	3.01993	9.54987	758.551	2.08929	4.50123	9.69762
9.13	83.3569	3.02159	9.55510	761.048	2.09005	4.50288	9.70116
9.14	83.5396	3.02324	9.56033	763.552	2.09081	4.50452	9.70470
9.15	83.7225	3.02490	9.56556	766.061	2.09158	4.50616	9.70824
9.16	83.9056	3.02655	9.57079	768.575	2.09234	4.50781	9.71177
9.17	84.0889	3.02820	9.57601	771.095	2.09310	4.50945	9.71531
9.18	84.2724	3.02985	9.58123	773.621	2.09386	4.51108	9.71884
9.19	84.4561	3.03150	9.58645	776.152	2.09462	4.51272	9.72236
9.20	84.6400	3.03315	9.59166	778.688	2.09538	4.51436	9.72589
9.21	84.8241	3.03480	9.59687	781.230	2.09614	4.51599	9.72941
9.22	85.0084	3.03645	9.60208	783.777	2.09690	4.51763	9.73293
9.23	85.1929	3.03809	9.60729	786.330	2.09765	4.51926	9.73645
9.24	85.3776	3.03974	9.61249	788.889	2.09841	4.52089	9.73996
9.25	85.5625	3.04138	9.61769	791.453	2.09917	4.52252	9.74348
9.26	85.7476	3.04302	9.62289	794.023	2.09992	4.52415	9.74699
9.27	85.9329	3.04467	9.62808	796.598	2.10068	4.52578	9.75049
9.28	86.1184	3.04631	9.63328	799.179	2.10144	4.52740	9.75400
9.29	86.3041	3.04795	9.63846	801.765	2.10219	4.52903	9.75750
9.30	86.4900	3.04959	9.64365	804.357	2.10294	4.53065	9.76100
9.31	86.6761	3.05123	9.64883	806.954	2.10370	4.53228	9.76450
9.32	86.8624	3.05287	9.65401	809.558	2.10445	4.53390	9.76799
9.33	87.0489	3.05450	9.65919	812.166	2.10520	4.53552	9.77148
9.34	87.2356	3.05614	9.66437	814.781	2.10595	4.53714	9.77497
9.35	87.4225	3.05778	9.66954	817.400	2.10671	4.53876	9.77846
9.36	87.6096	3.05941	9.67471	820.026	2.10746	4.54038	9.78195
9.37	87.7969	3.06105	9.67988	822.657	2.10821	4.54199	9.78543
9.38	87.9844	3.06268	9.68504	825.294	2.10896	4.54361	9.78891
9.39	88.1721	3.06431	9.69020	827.936	2.10971	4.54522	9.79239
9.40	88.3600	3.06594	9.69536	830.584	2.11045	4.54684	9.79586
9.41	88.5481	3.06757	9.70052	833.238	2.11120	4.54845	9.79933
9.42	88.7364	3.06920	9.70567	835.897	2.11195	4.55006	9.80280
9.43	88.9249	3.07083	9.71082	838.562	2.11270	4.55167	9.80627
9.44	89.1136	3.07246	9.71597	841.232	2.11344	4.55328	9.80974
9.45	89.3025	3.07409	9.72111	843.909	2.11419	4.55488	9.81320
9.46	89.4916	3.07571	9.72625	846.591	2.11494	4.55649	9.81666
9.47	89.6809	3.07734	9.73139	849.278	2.11568	4.55809	9.82012
9.48	89.8704	3.07896	9.73653	851.971	2.11642	4.55970	9.82357
9.49	90.0601	3.08058	9.74166	854.670	2.11717	4.56130	9.82703

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048
9.51	90.4401	3.08383	9.75192	860.085	2.11865	4.56450	9.83392
9.52	90.6304	3.08545	9.75705	862.801	2.11940	4.56610	9.83737
9.53	90.8209	3.08707	9.76217	865.523	2.12014	4.56770	9.84081
9.54	91.0116	3.08869	9.76729	868.251	2.12088	4.56930	9.84425
9.55	91.2025	3.09031	9.77241	870.984	2.12162	4.57089	9.84769
9.56	91.3936	3.09192	9.77753	873.723	2.12236	4.57249	9.85113
9.57	91.5849	3.09354	9.78264	876.467	2.12310	4.57408	9.85456
9.58	91.7764	3.09516	9.78775	879.218	2.12384	4.57567	9.85799
9.59	91.9681	3.09677	9.79285	881.974	2.12458	4.57727	9.86142
9.60	92.1600	3.09839	9.79796	884.736	2.12532	4.57886	9.86485
9.61	92.3521	3.10000	9.80306	887.504	2.12605	4.58045	9.86827
9.62	92.5444	3.10161	9.80816	890.277	2.12679	4.58204	9.87169
9.63	92.7369	3.10322	9.81326	893.056	2.12753	4.58362	9.87511
9.64	92.9296	3.10483	9.81835	895.841	2.12826	4.58521	9.87853
9.65	93.1225	3.10644	9.82344	898.632	2.12900	4.58679	9.88195
9.66	93.3156	3.10805	9.82853	901.429	2.12974	4.58838	9.88536
9.67	93.5089	3.10966	9.83362	904.231	2.13047	4.58996	9.88877
9.68	93.7024	3.11127	9.83870	907.039	2.13120	4.59154	9.89217
9.69	93.8961	3.11288	9.84378	909.853	2.13194	4.59312	9.89558
9.70	94.0900	3.11448	9.84886	912.673	2.13267	4.59470	9.89898
9.71	94.2841	3.11609	9.85393	915.499	2.13340	4.59628	9.90238
9.72	94.4784	3.11769	9.85901	918.330	2.13414	4.59786	9.90578
9.73	94.6729	3.11929	9.86408	921.167	2.13487	4.59943	9.90918
9.74	94.8676	3.12090	9.86914	924.010	2.13560	4.60101	9.91257
9.75	95.0625	3.12250	9.87421	926.859	2.13633	4.60258	9.91596
9.76	95.2576	3.12410	9.87927	929.714	2.13706	4.60416	9.91935
9.77	95.4529	3.12570	9.88433	932.575	2.13779	4.60573	9.92274
9.78	95.6484	3.12730	9.88939	935.441	2.13852	4.60730	9.92612
9.79	95.8441	3.12890	9.89444	938.314	2.13925	4.60887	9.92950
9.80	96.0400	3.13050	9.89949	941.192	2.13997	4.61044	9.93288
9.81	96.2361	3.13209	9.90454	944.076	2.14070	4.61200	9.93626
9.82	96.4324	3.13369	9.90959	946.966	2.14143	4.61357	9.93964
9.83	96.6289	3.13528	9.91464	949.862	2.14216	4.61514	9.94301
9.84	96.8256	3.13688	9.91968	952.764	2.14288	4.61670	9.94638
9.85	97.0225	3.13847	9.92472	955.672	2.14361	4.61826	9.94975
9.86	97.2196	3.14006	9.92975	958.585	2.14433	4.61983	9.95311
9.87	97.4169	3.14166	9.93479	961.505	2.14506	4.62139	9.95648
9.88	97.6144	3.14325	9.93982	964.430	2.14578	4.62295	9.95984
9.89	97.8121	3.14484	9.94485	967.362	2.14651	4.62451	9.96320
9.90	98.0100	3.14643	9.94987	970.299	2.14723	4.62607	9.96655
9.91	98.2081	3.14802	9.95490	973.242	2.14795	4.62762	9.96991
9.92	98.4064	3.14960	9.95992	976.191	2.14867	4.62918	9.97326
9.93	98.6049	3.15119	9.96494	979.147	2.14940	4.63073	9.97661
9.94	98.8036	3.15278	9.96995	982.108	2.15012	4.63229	9.97996
9.95	99.0025	3.15436	9.97497	985.075	2.15084	4.63384	9.98331
9.96	99.2016	3.15595	9.97998	988.048	2.15156	4.63539	9.98665
9.97	99.4009	3.15753	9.98499	991.027	2.15228	4.63694	9.98999
9.98	99.6004	3.15911	9.98999	994.012	2.15300	4.63849	9.99333
9.99	99.8001	3.16070	9.99500	997.003	2.15372	4.64004	9.99667

TABLE II — IMPORTANT NUMBERS

A. *Units of Length*

English Units		Metric Units	
12 inches (in.)	= 1 foot (ft.)	10 millimeters (mm.)	= 1 centimeter (cm.)
3 feet	= 1 yard (yd.)	10 centimeters	= 1 decimeter (dm.)
$5\frac{1}{2}$ yards	= 1 rod (rd.)	10 decimeters	= 1 meter (m.)
320 rods	= 1 mile (mi.)	10 meters	= 1 dekameter (Dm.)
		1000 meters	= 1 kilometer (Km.)

English to Metric	Metric to English
1 in. = 2.5400 cm.	1 cm. = 0.3937 in.
1 ft. = 30.480 cm.	1 m. = 39.37 in. = 3.2808 ft.
1 mi. = 1.6093 Km.	1 Km. = 0.6214 mi.

B. *Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

C. *Units of Measurement of Capacity*

Dry Measure		Liquid Measure	
2 pints (pt.)	= 1 quart (qt.)	4 gills (gi.)	= 1 pint (pt.)
8 quarts	= 1 peck (pk.)	2 pints	= 1 quart (qt.)
4 pecks	= 1 bushel (bu.)	4 quarts	= 1 gallon (gal.)
		1 gallon	= 231 cu. in.

D. *Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

E. *Other Numbers*

π = ratio of circumference to diameter of a circle
$= 3.14159265$
1 radian = angle subtended by an arc equal to the radius
$= 57° \, 17' \, 44''.8 = 57°.2957795 = 180°/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

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